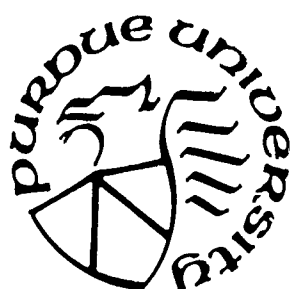


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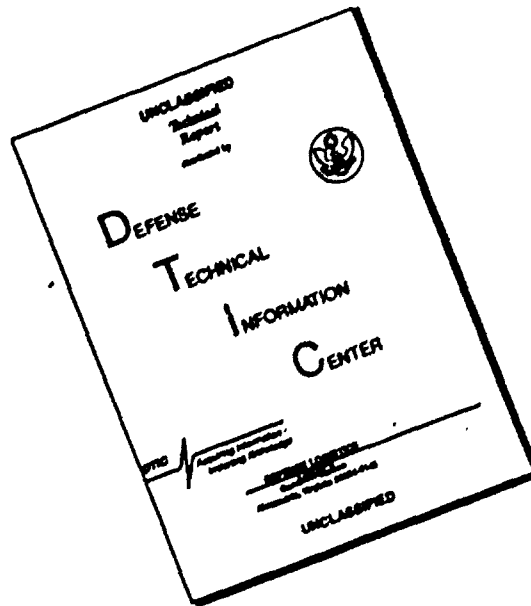


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DEPARTMENT OF STATISTICS

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DISTRIBUTIONS OF CHARACTERISTIC ROOTS

by

K. C. S. PILLAI
Purdue University

Department of Statistics
Division of Mathematical Sciences
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DISTRIBUTIONS OF CHARACTERISTIC ROOTS
IN MULTIVARIATE ANALYSIS^{1,2}

by

K.C.S. PILLAI
Purdue University

*New Words
and
Phrases:*

*Review, null and non-null distributions,
characteristic roots, test criteria, sub-asymptotic
and asymptotic expansions.*

ABSTRACT

A review of the work on distributions of characteristic (ch.) roots in real Gaussian multivariate analysis has been attempted surveying the developments in the field from the start covering about fifty years. The exact null and non-null distributions of the ch. roots have been reviewed and sub-asymptotic and asymptotic expansions of the distributions mostly for large sample sizes studied by various authors, have been briefly discussed. Such distributional properties of four test criteria and a few less important ones which are functions of the ch. roots have further been discussed in view of the power comparisons made in connection with tests of three multivariate hypotheses. In addition, one-sample case has also been considered in terms of distributional aspects of the ch. roots and criteria for tests of two hypotheses on the covariance matrix. A brief critical review has also been attempted. For convenience in organization, the review has been given in two parts: Part I. Null distributions and Part II. Non-null distributions.

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PART I NULL DISTRIBUTIONS

1. INTRODUCTION

In this part of the review, the null distribution of the ch. roots of sample matrices arising in three tests of hypotheses will be discussed first and then the one-sample case (Section 2). In Section 3, four criteria, namely, 1) Hotelling's trace, 2) Pillai's trace, 3) Roy's largest (smallest) root and 4) Wilks' criterion, recommended for tests of the three hypotheses and their optimal properties, and further three other less interesting criteria will be briefly treated. In addition, tests of two hypotheses on the covariance matrix will be considered in the one-sample case and also some test statistics in this connection. Section 4 consists of the study of the densities of individual ch. roots for which four different methods are treated. Distribution problems of the two trace criteria are studied in Section 5 discussing exact, approximate and asymptotic cases and showing the relation between the moments in the two cases. Section 6 contains a treatment of the distribution of Wilks' criterion, the multivariate beta distribution and associated independent beta variables. The exact and approximate distributions are considered. Other statistics of less importance are discussed in Section 7, namely, Wilks' statistic, Pillai's harmonic mean criteria and Bagai's statistic. Further, the one-sample case is taken up, treating the distribution problem of the likelihood ratio statistic for testing the covariance matrix equal to a specific matrix and those of sphericity, Wilks' sample generalized variance, ratios of ch. roots and elementary symmetric functions of the roots.

The review does not cover the material on complex Gaussian distributions although they are treated in a large number of papers listed in the References (For example, see James, 1964, Krishnaiah, 1976).

2. NULL DISTRIBUTION OF THE CHARACTERISTIC ROOTS

In multivariate statistical analysis often we wish to test one or more of the following three hypotheses: (Pillai, 1957, 1960)

- I) Equality of covariance matrices of two p-variate normal populations;
- II) Equality of the p-dimensional mean vectors of 2 p-variate normal populations having a common covariance matrix known as MANOVA (alternately general linear hypothesis); and
- III) Independence between a p-set and a q-set ($p \leq q$) in a (p+q)-variate normal population known as zero canonical correlation (ZCC).

Tests proposed for the above hypotheses are generally invariant tests (Roy, 1957, Lehmann, 1959) which have been shown to depend, under the null hypotheses, only on the characteristic roots of matrices based on sample observations. For example, in I), the tests depend on the characteristic roots of $S_1(S_1 + S_2)^{-1}$, where S_1 and S_2 denote the sum of product (SP) matrices with n_1 and n_2 degrees of freedom (df) and where both are almost everywhere positive definite (acpd). Thus $S_1(S_1 + S_2)^{-1}$ is acpd whence it follows that all the p ch. roots are greater than zero and less than unity. In II), the matrix is $S^*(S^* + S)^{-1}$, where S^* denotes the between SP matrix of means weighted by the sample sizes with 2-1 df and S denotes the within SP matrix (pooled from the SP matrices of 2 samples) with $N-2$ df where N is the total of the sizes of the 2 samples where S is acpd and S^* is at least positive semi-definite of rank $s = \min(p, 2-1)$. Thus, ac, s of the ch. roots are greater than zero and less than unity and p-s remaining roots are zero. In III), the matrix is $S_{11}^{-1}S_{12}S_{22}^{-1}S_{12}'$, where S_{11} is the SP matrix of the sample of observations on the p-set of variables, S_{22} , that on the q-set and S_{12} , the SP matrix between the observations on the p-set and those on the q-set. If $p \leq q$, $p+q < n'$, the sample size, then ac the p ch. roots of this matrix are greater than zero and less than unity.

In each of the three cases above, if the hypotheses to be tested is true, the $s \leq p$ non-zero ch. roots $b_i, i=1, \dots, s$, where $0 < b_1 \leq b_2 \leq \dots \leq b_s < 1$, have the same form of the joint density function which was independently obtained in 1939 by the five authors, Fisher (1939), Girshick (1939), Hsu (1939), Mood (1951) and Roy (1939). This joint density can be written in the form

$$(2.1) \quad f(b_1, \dots, b_s) = C(s, m, n) \prod_{i=1}^s b_i^m (1-b_i)^n \prod_{i>j}^s (b_i - b_j), \quad 0 < b_1 \leq \dots \leq b_s < 1,$$

where

$$(2.2) \quad C(s, m, n) = \pi^{\frac{s}{2}} \prod_{i=1}^s \{ \Gamma[\frac{1}{2}(2m+2n+s+1)] / \Gamma[\frac{1}{2}(2m+1+1)] \Gamma[\frac{1}{2}(2n+1+1)] \Gamma(i/2) \}.$$

Here m and n are to be understood differently for different situations. For example, in I),

$$(2.3) \quad m = \frac{1}{2}(n_1 - p - 1), \quad n = \frac{1}{2}(n_2 - p - 1).$$

In II),

$$(2.4) \quad m = \frac{1}{2}[(L-1)-p-1], \quad n = \frac{1}{2}[(N-L)-p-1].$$

In III),

$$(2.5) \quad m = \frac{1}{2}(q - p - 1), \quad n = \frac{1}{2}[(n' - 1 - q) - p - 1].$$

Note that m and n in (2.3) to (2.5) could be written in terms of v_1 and v_2 df, for example, in I), $v_1 = n_1, v_2 = n_2$; in II), $v_1 = L-1, v_2 = N-L$; and in III), $v_1 = q$ and $v_2 = n' - 1 - q$, where q is the df of the matrix $S_{12} S_{22}^{-1} S_{12}'$ and $n' - 1 - q$, of

$S_{11} - S_{12} S_{22}^{-1} S_{12}'$. Further, (2.2) may be rewritten in terms of v_1 and v_2 .

Alternately, there is interest in considering the ch. roots of $S_1 S_2^{-1}$, $S_2' S_1^{-1}$ or $(S_{12} S_{22}^{-1} S_{12}') (S_{11} - S_{12} S_{22}^{-1} S_{12}')^{-1}$ given by $f_i = b_i / (1 - b_i)$, $i=1, \dots, s$, where $0 < f_1 \leq \dots \leq f_s < \infty$. The joint density of f_1, \dots, f_s may be obtained from (2.1) in the form

$$(2.6) \quad g(f_1, \dots, f_s) = C(s, m, n) \prod_{i=1}^s [f_i^m / (1+f_i)] \prod_{i>j}^s (f_i - f_j)^{m+n+s+1}, \quad 0 < f_1 \leq \dots \leq f_s < \infty.$$

The form of the density in (2.1) will be called Type I and that in (2.6), Type II.

Further, if $g_i = nb_i$, $i=1, \dots, s$, as $n \rightarrow \infty$, the density of g_1, \dots, g_s is given by, (Nanda, 1948b)

$$(2.7) \quad h(g_1, \dots, g_s) = K(s, m) \prod_{i=1}^s (g_i^m e^{-g_i}) \prod_{i>j}^s (g_i - g_j), \quad 0 < g_1 \leq \dots \leq g_s < \infty,$$

where $K(s, m) = \pi^{\frac{s}{2}} / \prod_{i=1}^s \{ \Gamma[\frac{1}{2}(2m+i+1)] \Gamma(i/2) \}.$

The density (2.7) is of interest, especially under the hypothesis I) since S_1 has a Wishart distribution (Wishart, 1928) with v_1 df and covariance matrix Σ_1 , denoted by $W(p, v_1, \Sigma_1)$, the density of ch. roots of $S_1 \Sigma_1^{-1}$ when Σ_1 is known, could be written in the above form.

Again, it is easy to see that many of the commonly used distributions in statistics are special cases of (2.1) (or (2.6) or (2.7)). For example, if $p=1$ in (2.3), F distribution is obtained from (2.1) or more directly from (2.6) and chi-squared distribution from (2.7); if $k=2$ in (2.4), similarly is obtained from (2.6) the density of Mahalanobis' D^2 (Mahalanobis, 1930, 1936), (Hotelling's T^2 (Hotelling, 1931)), if $k=2$ and $p=1$ Fisher's z (student's t); if $p=1$ in (2.5), from (2.1) is obtained the density of the multiple correlation coefficient, if $p=1$, $q=1$ that of the simple correlation coefficient; and others.

An outline of the derivation of the distributions given above will be deferred to later sections (Section 6; Section 9, Lemma 9.1). For the derivation, in addition to the references preceding (2.1), references may be made to James (1954), Deemer and Olkin (1951) and Olkin and Roy (1954). In order to discuss the usefulness of these distributions it may be appropriate to consider briefly the important test statistics in the literature which are functions of the ch. roots. These will be considered in the next section.

3. SOME TEST CRITERIA

Four statistics which are commonly used for tests of the three hypotheses considered in the previous section are as follows:

- 1) Hotelling's trace, $U^{(s)} = \sum_{i=1}^s f_i$, (Pillai, 1954, 1955), with Hotelling's $T_0^2 = v_2 U^{(s)}$, (Hotelling, 1944, 1947, 1951);
- 2) Pillai's trace, $V^{(s)} = \sum_{i=1}^s b_i$, (1954, 1955);
- 3) Roy's largest (smallest) root, (1945, 1957); and
- 4) Wilks' criterion, $W^{(s)} = \prod_{i=1}^s (1-b_i)$, (1932).

The statistic 1) is known as Hotelling's trace and less frequently as Lawley-Hotelling trace. T_0^2 which was proposed by Hotelling in 1944 as a generalized T-test and measure of multivariate dispersion has been previously considered by several authors, Lawley (1938) as a generalization of Fisher's z -test, Bartlett (1939), and Hsu (1940a) who also obtained the null density for $s=2$ and the first and second non-central moments of the statistic explicitly. The statistic 2) known as Pillai's trace which was proposed by Pillai in 1954, (Pillai, 1954, 1955) was considered by Bartlett (1939), Hotelling (1947) and Nanda (1950). Wilks' criterion proposed in 1932 for test of hypothesis II)

is sometimes denoted by Wilks' Λ (older notation), U_{p, v_1, v_2} , and more recently by $W^{(s)}$ (see Pillai and Jayachandran, 1967, 1968) and is the oldest of the four test criteria but although the exact distribution was studied and percentage points obtained by Schatzoff (1966b), Pillai and Gupta (1969) and Lee (1972) only three decades later, a chi-squared approximation to its density suggested by Bartlett (1938, 1947a) and a beta approximation by Rao (1951, 1952) enabled the early use of the criterion. Further, Pillai (1955) proposed Wilks' criterion for test of hypothesis I) as well. The use of the other three criteria was facilitated by the pioneering statistical tables of Pillai (1957, 1960). The monotonicity of powers* of these criteria with respect to each population deviation parameter (see Section 9) has been studied by various authors (Das Gupta, Anderson and Mudholkar, 1964, Anderson and Das Gupta, 1964a, 1964b, Roy and Mikhail, 1961, Mikhail, 1962, Srivastava, 1964, Mudholkar, 1965, Perlman, 1974, Eaton and Perlman, 1974). For II), the admissibility of $U^{(s)}$ and the largest root has been established by Ghosh (1964) against unrestricted alternatives and Schwartz (1964) that of $V^{(s)}$ in the same sense. Kiefer and Schwartz (1965) have shown that $V^{(s)}$ test is admissible Bayes, fully invariant, similar and unbiased. They have also shown that $W^{(s)}$ is admissible Bayes, under a restriction, although admissibility could be established without this restriction. Giri (1968) has shown that $V^{(s)}$ is locally best invariant test for I) against one-sided alternatives. Pillai and Jayachandran (1967, 1968) have made exact power comparisons in the two-roots case of tests of each of the three hypotheses based on the four criteria and have shown that $U^{(2)}$, $V^{(2)}$ and $W^{(2)}$ compare favorably and behave somewhat in the same manner in regard to the three hypotheses but the largest root has lower power than the other three when there are more non-zero deviation parameters than one. Schatzoff (1966a) has similar findings in the case of II) in his monte carlo study. Again for II), Fujikoshi (1970) has obtained some approximate powers for $U^{(3)}$, $V^{(3)}$ and $W^{(3)}$ and Lee (1971b) some approximate powers for $s=3$ and 4, and Pillai and Sudjana (1974) exact powers for $U^{(3)}$ for all the three hypotheses and approximate powers for $s=3, 4$ and 5 for II).

* Although such aspects should be discussed under non-central distributions, this discussion is included here to give some idea of the relative importance of the different test criteria which would help the understanding of subsequent sections.

Exact power comparison of tests of I) based on individual roots have been made by Pillai and Al-Ani (1970) and for II) and III) by Pillai and Dotson (1969). Although the largest root has generally more power than the other roots, in small samples it appears that the smallest root or the middle root can have more power than the largest.

It may be noted that Wilks' criterion is a function of the likelihood ratio criterion (λ) for II) and III) but not for I). For I), the statistic $\prod_{i=1}^s b_i^{n_1} (1-b_i)^{n_2}$ is a function of the λ modified by Bartlett (1937) replacing sample size by df. The unbiasedness of the test has been shown by Sugira and Nagao (1968) against one-sided or two-sided alternatives while Wilks' criterion has been shown so far to have optimum properties only for one-sided alternatives. Again for two-sided alternatives Roy (1957) has shown the unbiasedness of the largest-smallest root test. No study has been made concerning $U^{(s)}$ and $V^{(s)}$ in the two-sided case. Some progress in this direction is made at Purdue University.

A few other statistics are available in the literature which are less important than any given above:

5) Wilks' statistic, $Z^{(s)} = \prod_{i=1}^s b_i$ (1932)

6) Pillai's harmonic mean, $H_1^{(s)} = (\sum_{i=1}^s b_i^{-1}/s)^{-1}$,

$H_2^{(s)} = (\sum_{i=1}^s (1-b_i)^{-1}/s)^{-1}$ and $H_3^{(s)} = (\sum_{i=1}^s f_i^{-1}/s)^{-1}$, (1955)

and 7) Bagai's statistic, $Y^{(s)} = \prod_{i=1}^s f_i$ (1962a).

Wilks' statistic $Z^{(s)}$, has been considered by (Gnanadesikan, Lauh, Snyder and Yao, 1965, Roy, Gnanadesikan and Srivastava, 1971). Hsu (1940a) has studied the moments and distribution problem of Wilks' statistic. The density of $Z^{(s)}$ has been obtained by Sudjana (1973b) and power comparisons made for the two-roots case with those of the four test criteria 1) to 4) for each of the three hypotheses I) to III). The results show that the test is generally behind in power to $U^{(2)}$, $V^{(2)}$ and $W^{(2)}$ but has greater power than the largest root. This general discouraging aspect of the power of the test was confirmed independently by Hart and Money (1976).

Approximations to the densities of the three harmonic mean criteria have been given by Pillai (1955). No adequate power studies have been carried out for the harmonic mean criteria or Bagai's statistic.

In the one-sample case i.e. in the case of a single covariance matrix, the following hypotheses are of great interest:

IV) $\underline{\mu} = \underline{\mu}_0$ (specified) in a p-variate normal population $N(\underline{\mu}, \underline{\Sigma})$.

V) $\underline{\Sigma} = \sigma^2 \underline{\Sigma}_0, \sigma^2 > 0$ unknown and $\underline{\Sigma}_0$ specified in $N(\underline{\mu}, \underline{\Sigma})$.

For IV), Roy (1957) has suggested the largest-smallest root test based on his union-intersection principle of hypothesis testing. Schuurmann and Waikar (1973) have studied the power function of the test in the bivariate case and shown that the test based on the acceptance region $(u^{-1} \leq \lambda_1 \leq \lambda_p \leq u) = 1-2\alpha$ where $\lambda_i = \lambda_{g_i}$ ($i=1, \dots, p$ with $s=p$) considered by Clemm, Krishnaiah and Waikar (1972) enjoys generally less power than the test studied by Thompson (1963) and Hanumara and Thompson (1968) whose acceptance region is given by $\Pr(\lambda_1 \leq \lambda_1 \leq \lambda_p \leq \lambda'') = 1-2\alpha, \Pr(\lambda_1 \leq \lambda') = 1-\alpha$. However the exact values of λ' and λ'' were given only for $p=2$ and approximate λ' and λ'' computed for $p \geq 3$ by taking separate lower $\alpha/2$ points for λ_1 and upper $\alpha/2$ points for λ_p so as to obtain a 2α level test. Further, some exact powers have been given by Sugiyama (1972a) for the bivariate case based on the largest root test alone and Muirhead (1974) has obtained approximate powers for the same test using asymptotic expansions against the alternate hypothesis $\underline{\Sigma} = \lambda \underline{I}$ for $p=2$ and 3.

A second test for IV) is based on the statistic $L = \frac{\prod_{i=1}^p \lambda_i^v - \sum_{i=1}^p g_i}{\prod_{i=1}^p g_i^v}$

(Anderson, 1958) is a function of λ and c where the g_i 's follow (2.7) with $s=p$ and v df. The exact distribution of L has been obtained by Nagarsenker and Pillai (1973) using inverse Mellin transform and percentage points tabulated for $p=2(1)10$. The unbiasedness of the L -test has been shown by Sugiura and Nagao (1968). The monotonicity of power function with respect to each of the p ch. roots of $\underline{\Sigma} \underline{\Sigma}_0^{-1}$ has been established by Nagao (1967) and Das Gupta (1969).

For hypothesis V), λ and c is a function of $W = \frac{\prod_{i=1}^p g_i}{(\sum_{i=1}^p g_i/p)^p}$ (Anderson, 1958)

which is the sphericity criterion of Mauchly (1940a,b) who obtained the null distribution for $p=2$. Nagarsenker and Pillai (1972, 1973) have obtained the exact null distribution of W in series form and tabulations of 5% and 1% points for $p=2(1)10$ made. The unbiasedness of the test has been shown by Gleser (1966), Sugiura and Nagao (1968). Muirhead (1976) has computed some powers in the two-roots case based on the distribution of W which he has obtained as mixtures of F distributions.

In the theory of principal component analysis (Hotelling, 1933, 1936) it is well known that the variances of the principal components are the ch. roots of the covariance matrix Σ . Since the maximum likelihood estimates of the ch. roots are the ch. roots of the mle of Σ , it is easy to see that the individual g_1 in (2.7) and its density are of added interest. Thus the next section is devoted mainly to the study of individual ch. roots from (2.1), (2.6) and (2.7).

4. INDIVIDUAL CHARACTERISTIC ROOTS

Four different methods have been developed over the years for the study of the distributions of individual ch. roots as follows: a) Roy-Pillai reduction formulae b) Mehta-Krishnaiah pdfs c) Davis differential equations (de's) and Pillai-Fukutomi-Sugiyama zonal polynomial series

4.1 ROY-PILLAI REDUCTION FORMULAE

Consider the integral over the domain $0 \leq x_1 \leq x_2 \leq \dots \leq x_k \leq x \leq 1$ of the function

$$(4.1) \quad \prod_{i=1}^k x_i^{q_i} (1-x_i)^r e^{tx_i} \prod_{i>j} (x_i - x_j),$$

where $q, r \geq -1$ and t is independent of the x 's. Let q_1, q_2, \dots, q_k be real numbers greater than -1 . Let us denote by $V(x; q_k, \dots, q_1; r; t)$ the pseudo-determinant (pseudo in the sense that the determinant has to be expanded preserving the order of integration)

$$(4.2) \quad \begin{vmatrix} \int_0^x x_k^{q_k} (1-x_k)^r e^{tx_k} dt_k & \dots & \int_0^x x_k^{q_1} (1-x_k)^r e^{tx_k} dx_k \\ \dots & \dots & \dots \\ \int_0^{x_2} x_1^{q_k} (1-x_1)^r e^{tx_1} dx_1 & \dots & \int_0^{x_2} x_1^{q_1} (1-x_1)^r e^{tx_1} dx_1 \end{vmatrix}.$$

Then it is easy to see that the integral of (4.1) can be expressed as (4.2) with $q_j = q + j - 1$ ($j=1, \dots, k$). Pillai (1954, 1956b) has proved the following theorem:

Theorem 4.1. The pseudo-determinant

$$(4.3) \quad V(x; q_k, q_{k-1}, \dots, q_1; r; t) = (q_k + r + 1)^{-1} (A^{(k)} + B^{(k)} + q_k C^{(k)} + t D^{(k)}),$$

where $A^{(k)} = -I_0(x; q_k; r+1; t) V(x; q_{k-1}, \dots, q_1; r; t),$

$$B^{(k)} = 2 \sum_{j=k-1}^1 (-1)^{k-j-1} I(x; q_k + q_j; 2r+1; 2t) V(x; q_{k-1}, \dots, q_{j+1}, q_{j-1}, \dots, q_1; r; t),$$

$$C^{(k)} = V(x; q_{k-1}, q_{k-1}, \dots, q_1; r; t),$$

$$D^{(k)} = V(x; q_k, \dots, q_1; r; t) - V(x; q_{k+1}, q_{k-1}, \dots, q_1; r; t),$$

$$I_0(x'; a; b; t) = x^a (1-x)^b e^{tx} \Big|_0^{x'} \text{ and } I(x'; a; b; t) = \int_0^{x'} x^a (1-x)^b e^{tx} dx.$$

As a special case of Theorem 4.1 (a second special case will be shown later for the mgf of $V^{(s)}$) putting $t=0$, $q_j = m+j-1$, $j=1, \dots, s$, the following reduction formula for the cdf of the largest root b_s is obtained: (Pillai, 1954, 1956b)

$$(4.4) \quad (m+n+s)V(x; m+s-1, \dots, m; n; 0) = -I_0(x; m+s-1; n+1; 0)V(x; m+s-2, \dots, m; n; 0) \\ + 2 \sum_{j=s-1}^1 (-1)^{s-j-1} I(x; 2m+s+j-2; 2n+1; 0)V(x; m+s-2, m+j, m+j-2, \dots, m; n; 0).$$

Using (4.4), Pillai (1954, 1956a) has obtained explicit expressions for the cdf of b_s ($s=1, \dots, s$) in terms of incomplete beta function. However, for larger values of s computation is extremely prohibitive and for this reason an approximation of the cdf of the largest root at the upper end has been suggested by Pillai (1954, 1956a, 1965a). This approximation will be discussed later.

The problem of obtaining the cdf of the largest root has been investigated first by Roy (1945, 1957) who gave explicit expressions for its cdf for $s=2, 3$ and 4. Nanda (1948a) also gave such expressions for $s=2(1)5$, but no general expression. Pillai and Dotson (1969) has proved the following lemma:

Lemma 4.1. If b_i and b_{s-i+1} are the i^{th} and $(s-i+1)^{\text{th}}$ roots ($i=1, \dots, s$) where b_1, \dots, b_s follow the density (2.1), then

$$(4.5) \quad \Pr(b_i \leq x; m, n) = 1 - \Pr(b_{s-i+1} \leq 1-x; n, m),$$

where on the right hand side of (4.5) m and n are interchanged.

Roy (1945) has also given an expression for the cdf of the i^{th} root ($i=1, \dots, s$). However the expression is not correct and even the total probability for the whole range does not equal unity. Pillai and Dotson (1969) have given the correct reduction formulae for the smallest root, second largest and second smallest roots. (Formula (4.5) is not useful when computations for larger values of m cannot be easily made as is usually the case.) They have also given upper 5% and 1% points for the smallest root for $p=2$ and 3 and middle root for $p=3$.

Returning to the problem of the extreme roots, Pillai (1965a) has obtained general expressions approximating at the upper end the cdf of the largest of s non-null ch. roots. These expressions are the only source for computing the cdf of the largest ch. root for larger values of s , and are given below.

(i) s even. The approximation to $\Pr(b_s \leq x)$ for even values of s is given by

$$(4.6) \quad 1 + \sum_{i=1}^{s-1} (-1)^i k_{m+s-i} I_0(x; m+s-i; n+1; 0), \quad k_{m+s} = 0, \text{ where}$$

$$k_{m+s-i} = (m+n+s-i+1)^{-1} [C(s, m, n) V(1; m+s-1, \dots, m+s-i+1, m+s-i-1, \dots,$$

$$m; n; 0) - (m+s-i+1) k_{m+s-i+1}],$$

$$C(s-1, m, n) V(1; m+s-1, \dots, m+s-i+1, m+s-i-1, \dots, m; n; 0) = \left[\binom{s-1}{i-1} \prod_{j=1}^{s-1} \frac{(2m+s-j+1)}{(2m+2n+2s-j+1)} \right].$$

(ii) s odd. The approximation to $\Pr(b_s \leq x)$ for odd values of s is given by

$$(4.7) \quad \frac{I(x; m; n; 0)}{B(m+1, n+1)} + \sum_{i=1}^{s-1} (-1)^i k_{m+s-i} I_0(x; m+s-i; n+1; 0).$$

The approximation neglects terms involving $(1-x)^{rn+1}$, $r > 1$, in the development of the cdf obtained by repeated application of (4.3). Pillai (1954, 1956a, 1957, 1960) gave the upper 5 and 1% points of b_s for $s=2$ and 5 using his approximation formulae. Sen (1957) computed similar upper percentage points for three roots and Ventura (1957) for four, both following Pillai's method. Pillai and Bantegui (1959) gave such tabulation for $s=6$. All these percentage points were given for values of $m=0(1)4$ and n varying from 5 to 1000. Further Jacildo (1959) extended the tables for $s=2$ and $s=3$ for values of $m=5, 7, 10$ and 15 and the same range of values of n as before. Pillai (1960) has published all these percentage points for $s=2(1)6$. Further, Pillai (1964a) has given such percentage points for $s=7$ for values of $m=0(1)5, 7, 10$ and values of n as before. For all the computations up to $s=7$ the approximation was obtained for each s since the general expressions (4.7) and (4.8) were obtained only in 1965 when percentage points were also given for $s=8, 9$ and 10 for values of $m=0(1)5, 7, 10, 15$ and n as before (1965a). Similar percentage points for $s=11$ and 12 are given in Pillai, (1970) and for $s=14(2)20$ in (1967b). Further, percentage points for $s=2(1)10$ and $s=13(1)20$ are available in two mimeograph reports in the Department of Statistics at Purdue University (Pillai, 1966b, No.76, 1966a, No.72).

Nanda (1951) has given upper 5 and 1% points of the largest root for $s=2$ and very small values of m and n ($m=0(\frac{1}{2})2; n=\frac{1}{2}(\frac{1}{2})10$). Foster and Rees (1957) have tabulated the percentage points (80, 85, 90, 95 and 99) of the largest root for

$s=2$ $m=-0.5$, $0(1)9$ and $n=1(1)19(5)49,59,79$. Foster (1957, 1958) has further extended these tables for values of $s=3$ and 4. The arguments they have used for tabulation are the degrees of freedom. Heck (1960) has given some charts of upper 5, 2.5 and 1% points for $s=2(1)5$, $m=-\frac{1}{2}$, $0(1)10$ and $n \geq 5$.

In the one sample case reduction formulae for individual roots could be obtained from these discussed above by the method used for obtaining (2.7) from (2.1). For example, for the largest root the reduction formula for the one sample case can be obtained from (4.3) in the following manner:

Let $W(y; q_k, \dots, q_1; -1)$ denote the pseudo-determinant obtained from (4.2) by putting $r=0$, $t=-1$ and x_i changed to y_i ($i=1, \dots, k$) and x to y where $0 < y_1 \leq \dots \leq y_k < \infty$, then

$$(4.8) \quad W(y; q_k, \dots, q_1; -1) = -I_0(y; q_k; -1)W(y; q_{k-1}, \dots, q_1; -1) \\ + 2 \sum_{j=k-1}^1 (-1)^{k-j-1} I(y; q_k + q_j; -2)W(y; q_{k-1}, \dots, q_{j+1}, q_{j-1}, \dots, q_1; -1) \\ + q_k W(y; q_k^{-1}, q_{k-1}, \dots, q_1; -1),$$

where $I(y; q; t) = \int_0^y y_1^q e^{ty_1} dy_1$ and $I_0(y; q; t) = y_1^q e^{ty_1} \Big|_0^y$.

Hence the cdf of g_s can be obtained from that of b_s and similarly those of other individual roots. Nanda (1948b) has given explicit expressions for the cdf of the largest root g_s for $s=2(1)5$ and similarly the smallest root and middle root for $s=3$. However since the computation is prohibitive for larger values of s Pillai and Chang (1968, 1970) and independently Hanumara and Thompson (1968) obtained an approximation to the cdf of g_s (approximating at the upper end) starting from Pillai's approximation for the cdf of b_s in (4.6) and (4.7) and using the method for deriving (2.7) from (2.1). The approximation is as follows:

(i) s even. The approximation of $\Pr(g_s \leq y)$ for even s is given by

$$(4.9) \quad 1 + \sum_{i=1}^{s-1} (-1)^i k_{m+s-i}^* y^{m+s-i} e^{-y}, \quad k_{m+s}^* = 0$$

where $k_{m+s-i}^* = \frac{K(s, m)}{K(s-1, m)} \binom{s-1}{i-1} \left(\frac{1}{2}\right)^{i-1} \prod_{j=1}^{i-1} (2m+s-j+1) - (m+s-i+1) k_{m+s-i+1}^*$.

(ii) s odd. The approximation to $\Pr(g_s \leq y)$ for odd s is given by

$$(4.10) \quad \frac{1}{\Gamma(m+1)} \int_0^y g^m e^{-g} dg + \sum_{i=1}^{s-1} (-1)^i k_{m+s-i}^* y^{m+s-i} e^{-y}.$$

Pillai and Chang have tabulated (1968, 1970) upper 10, 5, 2.5, 1 and 0.5% points of g_s for $s=2(1)20$ and df ranging from 2 to 200. Hanumara and Thompson (1968)

have tabulated upper percentage points as above except 10% for $s=2(1)10$ and also lower percentage points for the smallest root derived in view of lemma (4.1). Earlier, Thompson (1962) had tabulated upper percentage points for $s=2$.

4.2 MEHTA-KRISHNAIAH PFAFFIAN METHOD

Mehta (1960, 1967) has developed an alternate method of reduction of the pseudo-determinant in terms of double integrals which was extensively used with extensions by Krishnaiah and associates (Krishnaiah and Chang, 1970, 1971b, Krishnaiah and Waikar, 1970, 1971b). The reduction is as follows: Let

$$(4.11) \quad \mathcal{V}(x', x''; q_1, \dots, q_k; \psi) = \int_{x' \leq x_1 \leq \dots \leq x_k \leq x''} \dots \int V_0(x_1, \dots, x_k; q_1, \dots, q_k) \prod_{i=1}^k (\psi(x_i) dx_i),$$

where $V_0(x_1, \dots, x_k; q_1, \dots, q_k) = |(x_j^{q_i})_{i,j=1,\dots,k}|$. Further, let

$$F_T^t(x', x'') = \int_{x'}^{x''} F_T(x', u) u^t \psi(u) du, \quad F_T(x', u) = \int_{x'}^u v^r \psi(v) dv.$$

and $f_T^t(x', x'') = F_T^t(x', x'') - F_T^r(x', x'')$. In addition, let

$$D(x', x''; q_1, \dots, q_{2f}; \psi) = |(f_{q_i}^{q_j}(x', x''))_{i,j=1,\dots,2f}|^{\frac{1}{2}}$$

$$\text{and } G_t(x', x''; q_1, \dots, q_{2f+1}; \psi) = |(f_{q_i}^{q_j}(x', x''))_{i,j=1,\dots,t-1,t+1,\dots,2f+1}|^{\frac{1}{2}}.$$

Note that both D and G_t are pfaffians. (If T is a skew symmetric matrix of even order, $|T|^{\frac{1}{2}}$ is called a pfaffian.)

Lemma 4.2. Let $\psi(x)$ be a function such that the integral given in (4.11) exists and let $x' < x''$ be real constants. Then

$$(4.12) \quad \mathcal{V}(x', x''; q_1, \dots, q_k; \psi) = D(x', x''; q_1, \dots, q_{2f}; \psi) / f! \text{ when } k=2f$$

$$(4.13) \quad = \sum_{i=1}^{2f+1} (-1)^{i+1} F_{q_i}^{i+1}(x', x'') G_i(x', x''; q_1, \dots, q_{2f+1}; \psi) \text{ when } k=2f+1.$$

Mehta (1960) proved (4.12) in the special case when $q_i = i-1, i=1, \dots, k$ and $\psi(x) = e^{-x^n} x^r$. Krishnaiah and Chang (1970, 1971b) proved the above lemma using Mehta's method (1960) for the case when $q_i = i-1, i=1, \dots, q$. Lemma (4.2) is due to Krishnaiah and Waikar (1970, 1971b). It is obvious that (4.11) multiplied by appropriate coefficients include as special cases the cdf's of the largest root from (2.1) and (2.7).

Now in regard to the j^{th} root, $x_j, i \leq j < k$,

$$(4.14) \quad \Pr(x_j \leq x) = \Pr(x_{j+1} \leq x) + \Pr(x_1 < \dots < x_j < x < x_{j+1} < \dots < x_k)$$

which gives a recurrence formula for the cdf of intermediate roots. Once the cdf of the largest root is known it is enough to evaluate the second term on the right side of (4.14). (A similar approach knowing the cdf of the smallest root first is also available.) Now developing $V_0(\cdot)$ in (4.11) by the first j columns (noting that in the densities (2.1) and (2.7) $q_i = i-1$) and using lemma 4.2 to evaluate the integral

$$(4.15) \Pr(x_j < x) = \Pr(x_{j+1} < x) + c \sum_{\mathbf{q}} \psi(a, x; q'_1, \dots, q'_r; \psi) \psi(x, b; q'_{r+1}, \dots, q'_k; \psi),$$

where (a, b) is the range of the variables and c the constant in the joint density in question and $q'_1 < \dots < q'_r$ is a subset of the integers $0, 1, \dots, k-1$ and $q'_{r+1} < \dots < q'_k$ the complement set, $\sum_{\mathbf{q}}$ denotes summation over all $\binom{k}{r}$ possible choices of $q'_1 < \dots < q'_r$. Further, Krishnaiah and Waikar (1970, 1971b) have obtained the joint density of any subset of consecutive ordered roots in a similar manner.

Based on the joint distribution of the largest and smallest roots b_1 and b_s obtained by Krishnaiah and Chang (1970, 1971b) by methods described above, Schuurmann Waikar and Krishnaiah (1973) have obtained values of x such that $\Pr(1-x \leq b_1 \leq b_s \leq x) = 1-\alpha$ for $\alpha = 0.10, 0.05, .025, 0.01$; $m=0(1)5, 7, 10, 15$; $n=5(1)10(2)20(5)50$ and $s=2(1)10$. Since optimal choice of x' and x'' is not known, the authors carried out these tabulations for facilitating Roy's two-sided test of hypothesis I). Tables for the upper percentage points of b_1 has also been given by Schuurmann and Waikar (1972). Further Clemm, Krishnaiah and Waikar (1972) have tabulated the values of u such that $\Pr(u^{-1} \leq \lambda_1 \leq \lambda_s \leq u) = 1-\alpha$ where $\lambda_i = 2g_i (i=1, \dots, s)$ for values of $\alpha = 0.05, 0.025, 0.01, 0.005$, $s=2(1)10(2)20$, and $df = (p+1)(1)20(2)30(5)50$. This again has been to facilitate Roy's two-sided test of hypothesis IV). Thompson (1962) had computed the same for $s=2$. They have also provided various lower percentage values of λ_s . Further, upper percentage points of the individual roots $\lambda_i = 2g_i (i=1, \dots, s-1)$ for $s=2(1)10$ and the above values of df have been given by Clemm, Chattopadhyay and Krishnaiah (1972).

4.3 DAVIS DE METHOD

A third approach is due to Davis (1972a, 1972b) who has shown that the marginal density functions of b_i 's (g_i 's) satisfy a system of ordinary differential equations of Fuchsian type. Let $v_1, v_2 \geq p$ in (2.1) so that $s=p$. For notational convenience let $c_i = 1 - b_i (i=1, \dots, p)$. Further define $D^T(t, c) = \{0 < x_T < \dots < x_t < c < x_{t-1} < \dots < x_1 < 1\}$. Then the marginal density function $f_t(c)$ of c_t is given by

$$f_t(c) = \int_{[0,1]^{p-1}} \phi_{p,v_2,v_1}(x_1, \dots, x_{t-1}, c, x_t, \dots, x_{p-1}) dx$$

where $dx = \prod_{i=1}^{p-1} dx_i$, it is proportioned to

$$\frac{c^{v_2-p-1}}{(1-c)^{v_1-p-1}} \int_{[0,1]^{p-1}} \phi(x) \prod_{i=1}^{p-1} (c-x_i) dx,$$

where ϕ denotes ϕ_{p-1,v_2-1,v_1-1} and ϕ_{p,v_2,v_1} denotes the joint density of the

c_i 's. Again, define

$$\psi_r(c, x) = \phi(x) \sum_{\alpha} (c-x_{\alpha(1)}) \dots (c-x_{\alpha(p-1-r)}), (r=0, 1, \dots, p-1),$$

the summation being extended over the $\binom{p-1}{r}$ selections of integers

$\alpha(1) < \dots < \alpha(p-1-r)$ from the set $1, 2, \dots, p-1$. When $r = p-1$, the sum is taken to be unity. Now introduce the p functions

$$L_{t,r}(c) = \int_{[0,1]^{p-1}} \psi_r(c, x) dx, (r=0, 1, \dots, p-1).$$

The system of differential equations derived by Davis is the following:

$$(4.16) \quad c(1-c)L'_{t,r} = \frac{1}{2}(p-r)(v_1+v_2-p+r-1)L_{t,r-1} \\ + \frac{1}{2}r[(1-c)(v_2-p+r)-c(v_1-p+r)]L_{t,r} \\ + \frac{1}{2}(r+1)(r+2)c(1-c)L_{t,r+1}, (r=0, 1, \dots, p-1),$$

where $L_{t,-1} = L_{t,p} = 0$. Further, it is convenient to introduce $H_r = (1-c)^r L_{t,r}$ ($r=0, \dots, p-1$) and to express (4.16) as a matrix de for $H = (H_0, \dots, H_{p-1})$

given by

$$(4.17) \quad dH/dc = [c^{-1}A + (1-c)^{-1}C]H,$$

where matrices A and C are given in (1972a). The de (4.17) is of the Fuchsian type with regular singularities at $c=0, 1$ and ∞ . The marginal density of the largest root has been solved as a power series which coincides with that obtained by Pillai (1967a) and independently by Fukutomi and Sugiyama (1967). Pillai's approximation to the distributions of the largest (smallest) root in (4.6) and (4.7) are interpreted as exact solutions the contributions of higher order solutions being neglected. No approximations of a useful nature have been suggested for the distributions of the other ch. roots. Further, Davis (1972b) has given explicit expressions of the marginal distributions of individual roots b_i 's for $p \leq 5$. A de is applied recursively to construct the distributions for p from those obtained from $p-1$. Results also have been derived for g_i 's = $(2g_i$'s). The results for f_i 's in (2.6) would follow from those of b_i 's.

Finally the system of de's discussed above is related to Davis' de's for Hotelling's trace (Davis, 1968, 1970a, 1970c) and those for Pillai's trace (1970b).

The work of Davis and earlier authors have been generalized in various directions by Eckert (1975) where the ψ function in (4.11) includes the beta, gamma, normal and in addition, a class of ψ -functions given by weight functions for classical orthogonal polynomials to include Pearson system of curves distinguishing three more types. His reduction formulae generalize the work of Roy (1945, 1957), Nanda (1948a,b), Pillai and Dotson (1969) and Pillai (1954, 1956b) when $t=0$. He also generalizes the work of Davis (1972a, 1972b) in the context of the Pearson system. Further Eckert defines an operator to find all the cdf's of the individual roots, and in regard to the normal ψ finds the basic algebraic form for the distribution of the largest root in terms of a certain space of functions and one can pass on from the distribution of the largest root to those of the other roots through certain operators.

4.4 P-F-S ZONAL POLYNOMIAL SERIES

A fourth approach which has not been very useful in the null-case in view of the better approaches above but quite useful in the non-null case of the distribution problem, is the series approach involving zonal polynomials (James, 1961, 1964, Hua, 1959, Subrahmaniam, 1974, see Section 9) which are homogeneous symmetric polynomials in the ch. roots of a real symmetric matrix (Hermitian matrix in the complex case). The density of the largest root b_s has been obtained by Pillai (1967b) and independently by Fukutomi and Sugiyama (1967) as infinite series of zonal polynomials. The density of the smallest root follows from Lemma (4.1). Sugiyama (1967b) further gives the cdf of $b_s(b_1)$ as a series in powers of $x(1-x)$. The distributions of $f_s(f_1)$ follow by transformation used to obtain (2.6) from (2.1). For the one sample case, Sugiyama (1966, 1967a) has obtained the distribution of $\lambda_s = 2g_s$ and the density of the vector corresponding to λ_s useful in principal component analysis using some results of Tumura (1965), and has given an approximation (1972b) to the distribution of λ_s from the zonal polynomial view point. Further, Krishnaiah and Chang (1971a) have obtained the distribution of the smallest root λ_1 in terms of a finite series of zonal polynomials when m is a non-negative integer.

5. THE TWO TRACES

In this section, the two traces, those of Hotelling's and Pillai's, will be considered in some detail. For the study of the moment, Pillai (1954, 1956b) has obtained a recurrence relation concerning the mgf of $V^{(s)}$, which is a special case of (4.3) in Theorem 4.1 with $x=1$ and $q_j = m+j-1 (j=1, \dots, s)$ as given below.

$$(5.1) \quad (m+n+s-t)E(e^{tV^{(s)}}) + tE(V^{(s)}e^{tV^{(s)}}) = [2C(s, m, n)/C(s-2, m, n)] \\ \sum_{j=1}^{s-1} (-1)^{s-j-1} I(1; 2m+s+j-2; 2n+1; 2t) E[(\sum b_1, \dots, b_{s-j-1}) e^{tV^{(s-2)}}],$$

where $\sum b_1 \dots b_{s-j-1}$ denotes the $(s-j-1)^{th}$ esf in $(s-2)$ variables b_1, \dots, b_{s-2} .

Pillai studied the first four moments of $V^{(s)}$ using (5.1) and suggested a beta function approximation to the null distribution of $V^{(s)}$ (Pillai, 1954, 1955, 1957, 1960). Pillai and Mijares (1959) gave an alternate method of expressing the moments of $V^{(s)}$ and then the general expression for the fourth moment (Ting, 1959, Pillai, 1960) which was obtained by Pillai only for $s=2, 3$ and 4 earlier. The first four moments were used to obtain approximate upper 5% and 1% points of $V^{(s)}$ (Pillai, 1957, 1960) for $s=2(1)8$ and various values of m and n . (The method of using moment quotients to obtain percentage points is available in Pearson and Hartley (1956)). Further, Mijares (1964a) has tabulated the upper and lower 5% and 1% approximate percentage points of $V^{(s)}$ for $s=2(1)50$ using the moment quotients.

In regard to Hotelling's trace, as mentioned in the previous section, Pillai (1954, 1956b) has obtained a theorem similar to Theorem 4.1 concerning the pseudo-determinant for (2.6) in terms of the f_i 's from which a recurrence relation for the Laplace transform of $U^{(s)}$ corresponding to (5.1) follows as a special case. Pillai (1954, 1956b) has obtained the first four moments of $U^{(s)}$ and suggested an F approximation to the null distribution of the statistic. Pillai and Samson (1959) have obtained approximate upper 5% and 10% points for $s=2, 3$ and 4 using the moment quotients and Pillai (1957, 1960) has given such approximate percentage points for $s=2(1)8$ and various values of m and n . Further, Pillai (1964b) has obtained the following lemma which enables one to obtain the moments of $U^{(s)}$ from that of $V^{(s)}$ (and vice versa).

Lemma 5.1. Let $V_{i,m,n}^{(s)}$ and $U_{i,m,n}^{(s)}$ denote the i th elementary symmetric functions in the s g_i 's in (2.1) and s f_i 's in (2.6) respectively. Then the k^{th} moment $\mu_k\{U_{i,m,n}^{(s)}\}$ is derivable from $\mu_k\{V_{i,m,n}^{(s)}\}$ by making the following changes in the

expressions for the latter: (a) Multiply by -1 all terms except the term in n in each linear factor involving n and (b) change n to $m+n+s+1$ after performing (a).

Other approximations to the distribution of Hotelling's trace generally based on the moments have been given by Pillai and Young (1971) Tiku (1971) and Hughes and Saw (1972).

In regard to the exact distributions of the traces, there has been considerable work but no useful general forms have yet been obtained. Nanda (1950) has derived the cdf of $V^{(s)}$ in the special case of $m=0$ and $s=2$ and 3. The method was to study the mgf and carry out inversion involving the distributions of the weighted sums of two (or more) independent random variables of the form $(2n+k_i+1)(1-y_i)^{2n+k_i}$, $0 < y_i < 1$. Pillai and Jayachandran (1970) have extended the method further using (4.8) for reduction of pseudo-determinants arising in the derivation of the mgf and obtaining explicit expressions of the cdf of $V^{(s)}$ for $s=3$ and integral values of $m \leq 3$ and $s=4$ and $m=0$ and 1. Further, exact upper percentage points of $V^{(s)}$ have been computed for values of s and m given above and selected values of n . Earlier, Mikhail (1965) has obtained the exact density function of $V^{(2)}$.

In regard to Hotelling's trace, $U^{(s)}$, the distribution for $s=2$ has been obtained by Hsu (1940a) which has the same form as given by Hotelling (1951), in terms of an incomplete beta function. The density can be expressed in the form of a hypergeometric series. Pillai and Young (1971) and Pillai and Sudjana (1974) have developed the Laplace transform of $U^{(s)}$ as a pseudo-determinant, and using appropriate reduction formulae and inversions involving convolutions of two (or more) independent random variables (special type of F variables), obtained the exact distribution of $U^{(s)}$ for $p=3$ and 4 and small non-negative values of m .

Davis (1968) has shown that the density function of $U^{(s)}$ in the null case satisfies an ordinary linear differential equation of Fuchsian type and has computed percentage points (Davis, 1970c) by analytic continuation of Constantine's series distribution of $U^{(s)}$ (Constantine, 1966) which is convergent only for $|U^{(s)}| < 1$. Actually Constantine's series in the null case reduces to the relevant solution of the J_e in the unit circle about $U^{(s)}=0$. The tabulation is made of 5% and 1% points for $v_2 U^{(s)} / v_1$ for $s=3$ and 4 and various values of v_1 and v_2 . Pillai's approximate percentage points (1957, 1960) have been generally shown to have three decimal accuracy except for small values of v_2 .

Further, Davis (1970b) has shown that $V^{(s)}$ satisfies a kind of Fuchsian type which is related by the following simple transformation to his de for $U^{(s)}$. $U^{(s)} \rightarrow V^{(s)}$, $v_2 \rightarrow s - v_1 - v_2 + 1$. The transformation brings out the relationship between the moments of $U^{(s)}$ and $V^{(s)}$ (See Lemma 5.1 of Pillai). Again, Davis has found that Pillai's approximate percentage points of $V^{(s)}$ (Pillai, 1957, 1960) are accurate to four decimal places except when v_1 and v_2 are both small. Davis has also shown that the density $f_{v_1, v_2}(V^{(s)}) = f_{v_2, v_1}(s - V^{(s)})$ which can be used to obtain lower percentage points.

Again, Krishnaiah and Chang (1972) have studied the exact distributions of $V^{(s)}$ and $U^{(s)}$ using the pfaffian method discussed in the previous section to obtain the Laplace transforms as linear combinations of the products of certain double integrals. Further, for $V^{(s)}$ the double integrals have been developed by Schuurman, Krishnaiah and Chattopadhyay (1973) as linear combinations of incomplete gamma functions and inverted to obtain the density of $V^{(s)}$ explicitly for $s=2(1)6$ and several values of v_1 and v_2 . The authors have given the exact upper percentage points for $V^{(s)}$ for $s=2(1)5$, $\alpha=.01, .025, .05$ and $.10$ for m and $ns0(1)10(5)25$. They have confirmed the comments of Davis concerning Pillai's approximate percentage points. As regards $U^{(s)}$, Krishnaiah and Chang (1972) have considered explicitly only the case of $s=2$ and $v_1=5$.

In regard to asymptotic distributions of $V^{(s)}$ and $U^{(s)}$, asymptotic expansions for Hotelling's $T_0^2(=v_2 U^{(s)})$ both for the cdf and percentile have been given by Ito (1956) as a chi-squared series up to order v_2^{-2} the first term being $\chi_{pv_1}^2$. An independent derivation of Ito's expansion is given by Davis (1968, 1970a). Davis (1971) has extended Ito's expansions to order v_2^{-3} .

Muirhead (1970b) has considered a general asymptotic expansion for functions satisfying a certain system of partial de's, of which the distributions of $U^{(s)}$, $V^{(s)}$ and largest root in the one sample case fall as special cases. Muirhead explicitly obtains asymptotic expansions up to order v_2^{-2} for the cdf's of $V^{(s)}$ and the largest root. Further, Davis (1970b) has obtained an expansion for $V^{(s)}$ up to order v_2^{-3} .

Pillai (Pillai, 1973, Pillai and Sudjana, 1974) has suggested an F-type series form for $U^{(s)}$ which is exact for $s=2$ in the null case. The exactness has not been verified beyond $s=2$. The series involves zonal polynomials.

6. WILKS' CRITERION

Let \underline{A}_1 and \underline{A}_2 be positive definite symmetric matrices of order p , having independent Wishart distributions $W(p, v_i, \underline{E})$, $i=1,2$, respectively, (Anderson, 1958, Kshirsagar, 1972, Roy, 1957). Let $\underline{A}_1 = \underline{C}\underline{B}\underline{C}'$ and $\underline{A}_1 + \underline{A}_2 = \underline{C}\underline{C}'$, where \underline{C} is a lower triangular matrix. Then it has been shown (Hsu, 1939) that \underline{B} and \underline{C} are independently distributed and the density function of \underline{B} is given by (Khatri, 1959, Olkin and Rubin, 1964).

$$(6.1) \quad C_1(p, v_1, v_2) |\underline{B}|^{\frac{1}{2}(v_1 - p - 1)} |I - \underline{B}|^{\frac{1}{2}(v_2 - p - 1)}$$

$$\text{where } C_1(p, v_1, v_2) = \pi^{-p(p-1)/4} \prod_{i=1}^p \{ \Gamma[\frac{1}{2}(v_1 + v_2 - i + 1)] / \Gamma[\frac{1}{2}(v_1 - i + 1)] \Gamma[\frac{1}{2}(v_2 - i + 1)] \}.$$

The density (6.1) is known as the multivariate beta distribution. If \underline{B} is further diagonalized by an orthogonal transformation and integration is carried out with respect to the random variables in the orthogonal matrix, we obtain the joint density of the roots b_i 's in (2.1). The method can be modified suitably for the case of \underline{A}_1 being at least positive semi-definite. (See Section 9, eq (9.10)).

Again let $\underline{L} = I - \underline{B}$. Then Wilks' criterion $W^{(p)} = |\underline{L}|$. Wilks (1932) has shown through the uniqueness of the moments easily derivable from (2.1) that the density of $W^{(p)}$ could be written as a product of p independent beta variables. (See Theorems 8.5.1 and 8.5.2 of Anderson, 1958). Now let $\underline{L} = \underline{T}\underline{T}'$, where \underline{T} is a lower triangular matrix. Then $|\underline{L}| = \prod_{i=1}^p t_{ii}^2$, where t_{ii} is the i^{th} diagonal element of \underline{T} . Kshirsagar (1961) showed that t_{ii} ($i=1, \dots, p$) are independently distributed and that t_{ii}^2 follows the distribution

$$(6.2) \quad f_i(t_{ii}^2) = (t_{ii}^2)^{\frac{1}{2}(v_2 - i + 1) - 1} (1 - t_{ii}^2)^{\frac{1}{2}v_1 - 1} / \beta[\frac{1}{2}(v_2 - i + 1), \frac{1}{2}v_1], 0 < t_{ii}^2 < 1.$$

Further, Pillai (1964c, 1966c) has shown that the $p(p+1)/2$ distinct elements of \underline{L} can be transformed to $p(p+1)/2$ independent beta variables and since $V^{(p)} = \text{tr } \underline{B}$ and $U^{(p)} = \text{tr}(\underline{I} - \underline{B})^{-1} - p$, an alternate method of computing the moments of these statistics has been given using the independent beta variables. Again, Khatri and Pillai (1965) has given a method of obtaining the densities of the independent beta variables in successive stages.

In regard to Wilks' criterion, Wilks (1935) has obtained the exact null distribution in the form of a $(p-1)$ -fold multiple integral which he evaluated explicitly for $p=1,2$, for $p=3$ and $v_1=3,4$, and $p=4$ and $v_1=4$. Further,

asymptotic approximations have been considered by several authors. For example, Bartlett (1938, 1947a) obtained a chi-squared approximation to $-C \log W^{(p)}$ where $C = v_2 - \frac{1}{2}(p-v_1+1)$. Wald and Brookner (1941) developed an asymptotic expansion which was further modified by Rao (1948) to obtain the first three terms of a more rapidly convergent series for the cdf of $-C \log W^{(p)}$. Further, Box (1949) gave asymptotic approximations to functions of general lr statistics which include Rao's as a special case. But, in addition, Rao (1951, 1952) has given a second approximation as a beta series.

Schatzoff (1966b) has given a method for obtaining the exact distribution of $W^{(p)}$ using the representation of $-\log W^{(p)}$ as a sum of independent variables and taking successive convolutions. However, he has not given explicit expressions for the density or cdf of $W^{(p)}$ while Pillai and Gupta (1969) have given both for values of p up to 6. Schatzoff used Theorem 8.5.1 of Anderson while Pillai and Gupta used Theorem 8.5.2 (Anderson, 1958). Schatzoff has tabulated the factors for converting $\chi^2_{pv_2}$ percentiles to exact

percentiles of $-C \log W^{(p)}$ for $p=3(1)8$ and values of v_1 such that $pv_1 \leq 70$. Pillai and Gupta (1969) have lifted this restriction. Unlike Consul (1966c) who, following Nair (1938), has used inverse Mellin transform to obtain the distribution of $W^{(p)}$ for p up to 4 as infinite series, Pillai and Gupta (1969) have given the same in finite series form except when both p and v_1 are odd in which case the series is infinite. Further for $p=3(1)6$, they have extended Schatzoff's tables for selected values of $v_1 \geq 11$ and ≤ 22 .

Lee (1972) has studied the density and cdf of $W^{(p)}$ when p or v_1 is even with a view to simplifying further the expressions obtained by earlier authors for numerical computation. When p and v_1 are both odd, the expressions given are in terms of simple integrals. Asymptotic expansions also have been developed. Tables of the chi-squared correction factors have been extended to cover, in conjunction with previous tables, values of $p \leq v_1 \leq 20$ and $pv_1 \leq 144$ with omission when p or v_1 is odd and > 10 .

Mathai (1971a, b) has derived the null distribution using inverse Mellin transform, theory of residue and psi and zeta functions and computed upper 5% and 1% points for $W^{(p)}$ for selected values of the arguments. Mathai and Rathie (1971b) have obtained the exact null distribution of $W^{(p)}$ in the general case in terms of simple algebraic functions which can be computed without much difficulty. The density and the cdf are both given and some special cases are obtained from the general form which coincides with the results of Consul. The method of derivation again is taking the inverse Mellin transform. But before taking the transform of the ratios of the gamma products in the k th moment of $W^{(p)}$, elimination of the gammas by cancellation of common factors has been made and splitting the factors with the help of partial fraction methods has been carried out. Results are given separately for the four cases of p and v_1 taking odd and even values. Further, Bagai (1972a) has also expressed the null distribution of $W^{(p)}$ in terms of Meijer's G-function using inverse Mellin transform.

7. OTHER STATISTICS

In this section, the three statistics 5) to 7) of Section 3, namely, Wilks' statistic, Pillai's harmonic mean criteria and Bagai's statistic will be considered first and then statistics for the one-sample case.

Wilks' statistic, $Z^{(s)}$, considered by Hsu (1940a) and Lawley (1956) has been studied by Bagai (1964b) and Consul (1964b) obtaining the limiting distribution using inverse Mellin transform for $s=2(1)8$. Ito (1962) has discussed $Z^{(s)}$ as also Troskie (1966, 1972) and de Waal (1968, 1970). Pillai and Nagarsenker (1972) have recommended $Z^{(s)}$ for tests of hypotheses I) to III) and studied the general distribution problem. The density of $Z^{(s)}$ has been obtained in various forms by Sudjana (1973b) and further by Hart and Money (1976).

Harmonic mean criteria were proposed by Pillai in 1955 and obtained approximations to the densities in the three cases (Pillai, 1955). The exact distribution problem and the moments of the $H_i^{(s)}$ criteria have been studied by Troskie (1971, Troskie and Money, 1972) in terms of zonal polynomials.

Bagai's statistic, $Y^{(s)}$, has been studied by Bagai in several papers (1962a, 1964b, 1965b, 1967, 1972a, 1972b) giving exact and limiting distributions in integral forms first and then hypergeometric functions using

inverse Mellin transform. Explicit results are given for $s=2(1)8$ in the exact case and $s=5(1)10$ in the limiting case. Further, Consul (1964b) has obtained the limiting distribution for $s=2(1)8$, and exact distribution in integral form using inverse Mellin transform. Further, Mathai (1970a,b, 1972b) has obtained the exact distribution in terms of Meijer's G-function through inverse Mellin transform. Some results on series expansions of G-functions are also given. Further, Oliver (1972) has also considered the distribution problem in terms of the G-function.

In the one-sample case, the distribution of the L statistic for the test of IV) has been obtained by Nagarsenker and Pillai (1973) and percentage points given for $p=2(1)10$. Korin (1968) has expressed the null distribution of $-2\log L$ in the form of an asymptotic series of central chi-squared distributions and computed percentage points but his tables are incomplete for small v and $p=3(1)10$ (Pearson and Hartley (1972)). Davis (1971) has expressed the percentage points of $-2 \log L$ in terms of chi-squared percentiles using a Cornish-Fisher inversion of Box's series but his tables are also incomplete in regard to small values of v for the values of p he has considered i.e. $p=6$ and 10 . An asymptotic expansion of the distribution of $-2 \log L$ up to order v^{-3} has been given by Sugiura (1969b) involving chi-squared terms inverting characteristic function.

The sphericity criterion of Mauchly (1940a,b) for test of V) has been considered by several authors. The exact distribution of W has been obtained by Mauchly (1940b) for $p=2$ and by Consul (1967b) for $p=3,4$ and 6 and further by Consul (1969), Mathai (1970a,b, 1971b) and Mathai and Rathie (1970) and by John (1972) for $p=3$. The expression given by Consul (1969) is in terms of Meijer's G-functions (Meijer, 1946a, 1946b) where as those of Mathai and Rathie are in series form. Nagarsenker and Pillai (1972, 1973) have obtained the exact distribution of W in series form by methods developed similar to the one used by Box (1949) (also see Andersen, 1958) and Nair (1938, 1940) and tables of 5 and 1% points for $p=2(1)10$ made. Bagai (1965b, 1972a) has also obtained the null distributions of W as a multiple series. An asymptotic expansion of the distribution of the criterion $-2\rho \log W$ to order v^{-2} involving chi-square terms has been given by Anderson (1958, p.263), where ρ is a correction factor depending upon p and v .

Wilks (1932) defined the determinant of the covariance matrix as a scalar measure of scatter in a multivariate distribution and called it generalized variance. Wilks (1960) has discussed the relationship of

scatter to various other problems. Hence the distribution of the determinant of the SP matrix of a sample from a p -variate normal population has been studied by various authors. The ratio of the determinants of the SP matrix and the covariance matrix can be expressed as a product of p independent chi-squared variables. Wijsman (1957) has studied the distribution of sample generalized variance. Bagai (1962b, 1965a) has obtained the distribution in the null case (actually in the noncentral linear case) for $p=2(1)10$ expressing in terms of integrals. Consul (1964a) has also obtained the exact distribution in the null (linear) case using inverse Mellin transform and special cases given for $p=2(1)7$. Again Mathai (1970a) and Mathai and Rathie (1971a) have obtained the exact distribution in the null (linear) case using inverse Mellin transform and G-functions and expansions by calculus of residues, psi and zeta functions. Mathai (1970a) has given the distribution of the product of p independent gamma variables as a G-function and similarly that of the product of p independent beta variables. He deduces several interesting examples from these results including the generalized variance for the gamma case, Wilks' criterion and others for the beta case. Steyn (1967) and Oliver (1972) have also considered the distribution problem, the latter in terms of inverse Mellin transform.

Khatri (1967), and Pillai and Al-Ani (Pillai, Al-Ani and Jouris, 1969) have considered a hypothesis $\delta \Sigma_1 = \Sigma_2$, $\delta > 0$ unknown which includes I) as a special case and suggested the use of ratios of the ch. roots, b_i/b_j (or g_i/g_j in the limiting case) for tests in this connection. The latter authors have also obtained the distribution of the ratio of g_i/g_j for $s=2,3$ and 4. Krishnaiah and Waikar (1971a, 1972) have also obtained the distribution of the ratios of successive roots and each root to the maximum root in connection with simultaneous tests for equality of latent roots against certain alternatives in the one-sample, two-sample, MANOVA and canonical correlation cases. Further, Krishnaiah and Schuurmann (1974) have obtained the distribution of the ratios of the individual roots to the trace in the one-sample case and some percentage points obtained in connection with certain simultaneous tests of the hypothesis (Krishnaiah and Waikar, 1971a, 1972). In this connection use has been made of a relationship established by Davis (1972c) between the Laplace transforms of the ratios of individual roots to the sum of the roots and the densities of the above individual roots in the one-sample case.

Further, elementary symmetric functions (esf) of the ch. roots have been of some interest as statistics for the different tests of hypotheses of which Hotelling's and Pillai's traces are special cases. (See Lemma 5.1) Monotonicity properties of these criteria in connection with test 1) have been discussed by Anderson and Das Gupta (1964b). Mijares (1961) has shown some properties of completely homogeneous symmetric functions and certain determinantal results to give an inverse derivation of the moments of $V^{(s)}$. The method has been further extended to the moments in general of esf of the b_i illustrating explicitly for the second moment of $V_{2,m,n}^{(3)}$. Mijares (1964b) has obtained determinantal expressions for moments illustrating the results to third moment of second esf and product moments of first, second and third esf's. Pillai (1965b), in addition to Lemma 5.1, has shown some relations between the r^{th} moment of $U_{s-i,m,n}^{(s)}$ and $U_{i,n-r,m+r}^{(s)}$ as well as the r^{th} moment of $V_{s-1,m,n}^{(s)}$ and a linear function of the moments of order up to r of $U_{1,n,m+r}^{(s)}$ and obtained approximate upper 5 and 1% points for $U_{2,m,n}^{(3)}$ for various values of m and n . Such approximate percentage points have been obtained by Dotson (1968) for $V_{2,m,n}^{(3)}$. In the one sample case, the second esf has been studied by Pillai and Gupta (1967) in regard to first four moments, suggesting an approximation to its distribution.

PART II
NON-NULL DISTRIBUTIONS

8. INTRODUCTION

This part of the paper deals with the non-central distributions of the ch. roots in connection with the three hypotheses but as special cases of Pillai's distribution of the ch. roots under violations (Pillai, 1975). The exact, sub-asymptotic and asymptotic distributions of the different test statistics are considered and the one-sample case also treated. Section 9 reviews the distribution of the ch. roots under violations covering the three standard distributions and the one-sample case and in addition, treats the multivariate beta distribution. Further, sub-asymptotic distributions are discussed briefly in Section 10 and the distribution of individual ch. roots in Section 11 dealing with both exact and asymptotic cases. The distribution problem of Hotelling's trace is considered next in Section 12 discussing the moments, the exact distribution and asymptotic expansions. Section 13 surveys similar studies for Pillai's trace and Section 14 of Wilks' criterion. As in the null case, other statistics are treated in Section 15, namely, Wilks' statistic, $Z^{(p)}$, Pillai's harmonic mean criteria, Bagai's statistic, $\gamma^{(p)}$, and the \mathcal{L}_r test of equality of two covariance matrices. Actually, these criteria except the harmonic means fall under a class proposed by Pillai and Nagarsenker (1972). Finally, a short critical review is attempted in Section 16 which shows that in spite of all the efforts of hundreds of researchers in the field, the results achieved so far generally remain unsatisfactory for various reasons.

4. NON-NULL DISTRIBUTIONS OF THE CH. ROOTS

The joint distribution of the ch. roots has the same form (see Eq (2.1) or (2.6) or (2.7)), under the three null hypotheses I) to III). However, in the non-null case, the joint distributions are different in the three cases. James (1964) has listed them systematically and the reader may be referred to his paper for a close study. Nevertheless, in the next section, by a suitable approach, a single expression for the non-null distribution is given, of which the noncentral distributions for I) and II) fall as special cases and III) also follows. In this section, a few preliminaries will be discussed.

It has been shown (Roy, 1957) that the joint density of the ch. roots of $S_1 S_2^{-1}$ for I) involves as parameters only the ch. roots of $\Sigma_1 \Sigma_2^{-1}$ which will be denoted by $\underline{\Lambda} = \text{diag}(\lambda_1, \dots, \lambda_p)$. Such result is true for II) and III) as well with the respective parameter matrices $\frac{1}{2} \Sigma^{-1} \mu \mu'$ having ch. root matrix (say) $\underline{u} = \text{diag}(w_1, \dots, w_p)$ and $\Sigma_1^{-1} \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{12}'$ having ch. root matrix $P^2 = \text{diag}(p_1^2, \dots, p_p^2)$. Here $\frac{1}{2} \Sigma^{-1} \mu \mu'$ is the noncentrality matrix in the noncentral Wishart distribution of S^* for II) (Anderson, 1946, Anderson and Girshick, 1944, Weibull, 1953, James, 1955a, Herz, 1955, Constantine, 1963), and for III), the covariance matrix

$$\Sigma = \begin{pmatrix} p & \Sigma_{11} & \Sigma_{12} \\ q & \Sigma_{12}' & \Sigma_{22} \\ p & q & \end{pmatrix}. \quad (\text{Further, refer Roy, 1957, Anderson, 1958, to see how the test$$

of II) and that of general linear hypothesis lead to the same distribution problem).

The method employed for a unified approach to the distribution problem is to make the parameter matrix partially random (denoted "random" hereafter) which implies diagonalization by an orthogonal transformation H and integration over H ; in other words putting a Haar prior on H (see James, 1964, Constantine, 1963, Anderson, 1958, for Haar measure) leaving the latent roots of the parameter matrix non-random. Now a theorem is given below (Pillai, 1975) concerning the distribution of the characteristic roots of $S_1 S_2^{-1}$ when $S_1 (p \times p)$ has a non-central Wishart distribution with v_1 df and noncentrality matrix $\frac{1}{2} \Sigma_1^{-1} \mu \mu'$ and covariance matrix Σ_1 , and $S_2 (p \times p)$ has an independently distributed central Wishart distribution with v_2 df and covariance matrix Σ_2 , where $v_1, v_2 \geq p$. Let $\underline{F} = \text{diag.}(f_1, \dots, f_p)$ where f_1, \dots, f_p are the ch. roots of $S_1 S_2^{-1}$.

Theorem 9.1. The joint density of f_1, \dots, f_p is given by

$$(9.1) \quad C_2(p, v_1, v_2) e^{-\text{tr} \Lambda} |\Lambda|^{-\frac{1}{2}v_1} |F|^m |I| + \lambda F \prod_{i>j}^{-\frac{1}{2}(v_1+v_2)} (f_i - f_j) \\ \sum_{k=0}^{\infty} \sum_{\kappa} \left(\frac{1}{2}(v_1+v_2) \right)_{\kappa} C_{\kappa}(\lambda F(I + \lambda F)^{-1}) / k! \\ \sum_{d=0}^k \sum_{\delta} a_{\kappa, \delta} C_{\delta}(-\lambda^{-1} \Lambda^{-1}) L_{\delta}^m(\Lambda) / \left[\frac{1}{2} v_1 \right]_{\delta} C_{\delta}^2(I), 0 < f_1 \leq f_2 \leq \dots \leq f_p < \infty,$$

where Λ is "random", λ is a positive real number, $m = \frac{1}{2}(v_1 - p - 1)$ and

$$C_2(p, v_1, v_2) = \left[\prod_{i=1}^p \Gamma_p \left(\frac{1}{2}(v_1 + v_2) \right) \right] / \left[\Gamma_p \left(\frac{1}{2}v_1 \right) \Gamma_p \left(\frac{1}{2}v_2 \right) \Gamma_p \left(\frac{1}{2}p \right) \right],$$

where $\Gamma_p(a) = \pi^{\frac{1}{2}p(p-1)} \prod_{i=1}^p \Gamma(a - \frac{1}{2}(i-1))$. $C_{\kappa}(A)$ is the zonal polynomial of

degree k corresponding to the partition $\kappa = (k_1, \dots, k_p)$ of k into not more than p parts, $k_1 \geq k_2 \geq \dots \geq k_p \geq 0$, $k_1 + \dots + k_p = k$. The generalized Laguerre polynomial $L_{\kappa}^m(\Lambda)$ is defined in Eq (14) of Constantine (1966) and $a_{\kappa, \delta}$ are constants (see Eq (20) of Constantine (1966), given in (9.3) below, tabulations in Pillai and Jouris (1969)). Further, $(a)_{\kappa} = \prod_{i=1}^p (a - \frac{1}{2}(i-1))_{k_i}$ with $(a)_k = a(a+1)\dots(a+k-1)$.

It may be noted that $C_2(p, v_1, v_2) = C(p, m, n)$ given in the null case and $\Gamma_p(a)$ is proportioned to the value of the Wishart integral having $2a$ df with $\Sigma = I$. Zonal polynomial $C_{\kappa}(A)$ is a homogeneous symmetric polynomial in the ch. roots of the symmetric matrix A such that $(\text{tr } A)^k = \sum_{\kappa} C_{\kappa}(A)$ and the value of $C_{\kappa}(I)$ is given in Eq (21) of James (1964). James (1961, 1964) has defined $Z_{\kappa}(A)$ by the relation $C_{\kappa}(A) = [\chi_{[2\kappa]}(1)] [2^k k! / 2k!] Z_{\kappa}(A)$ where $\chi_{[2\kappa]}(1)$ is the dimension of the representation $[2\kappa]$ of the symmetric group on $2k$ symbols (see Eq (19) of James (1964)). James and Parkhurst have tabulated coefficients of the zonal polynomials expressed both in terms of elementary symmetric functions of the ch. roots and in terms of sums of powers for $k=1(1)12$ and are given in Harter and Owen (1974). $Z_{(1^k)}(A) = k! a_k$, where a_k is the k^{th} esf in the ch. roots of A and orthogonality relations to obtain other zonal polynomials of degree k are given in James (1964). James (1968) has also given an alternate method of construction of the zonal polynomials by use of the Laplace-Beltrami operator. Crowther and Young (1974) have suggested a third method of construction. Further,

$$(9.2) \quad L_{\kappa}^m(\Lambda) [(\frac{1}{2}v_1)_{\kappa} C_{\kappa}(I)] = \sum_{t=0}^k \sum_{\tau} (-1)^t a_{\kappa, \tau} C_{\tau}(\Lambda) / [(\frac{1}{2}v_1)_{\tau} C_{\tau}(I)].$$

The $a_{\kappa, \tau}$ coefficients have been studied in addition to Constantine (1966) and Pillai and Jouris (1969), by Bingham (1974), Muirhead (1974) and others.

The density of b_1, \dots, b_p can be obtained from (9.1) by the transformation $b_i = \lambda f_i / (1 + \lambda f_i), i=1, \dots, p$. Formula (9.1) yields the following special cases:

(a) For $\Omega=0$, from Eq (21) of Constantine (1966), (see also Herz (1955) page 487), $L_{\kappa}^m(0) = (\frac{1}{2}v_1)_{\kappa} C_{\kappa}(I)$. Substituting $L_{\kappa}^m(0)$ into (9.1) and making use of (Constantine, 1966)

$$(9.3) \quad C_{\kappa}(I-\Lambda)/C_{\kappa}(I) = \sum_{t=0}^k \sum_{\tau} (-1)^t a_{\kappa, \tau} C_{\tau}(\Lambda)/C_{\tau}(I),$$

the non-null distribution of Khatri (1967) for I) is obtained.

b) By letting $\Lambda=I$ and $\lambda=1$ and by using the relation (9.2) with $\tau=\delta=\kappa$, the non-central distribution of Constantine (1963) for II) is deduced.

Further, if Ω in (9.1) is considered a completely random matrix $\frac{1}{2} \Sigma_1^{-\frac{1}{2}} M Y Y' M' \Sigma_1^{-\frac{1}{2}}$ where $M(p \times q)$ is a parameter matrix and $Y Y'$ has a central Wishart distribution $W(q, v_3, \Sigma_3)$, then we get the following theorem: (Pillai, 1975)

Theorem 9.2. The joint density of f_1, \dots, f_p , where Ω is completely random, is given by

$$(9.4) \quad C_2(p, v_1, v_2) |\Lambda|^{-\frac{1}{2}v_1} |I + \Omega_1|^{-\frac{1}{2}v_3} |F|^m |I + \lambda F|^{-\frac{1}{2}(v_1+v_2)} \prod_{i>j} (f_i - f_j) \\ \sum_{k=0}^{\infty} \sum_{\kappa} [(\frac{1}{2}(v_1+v_2))_{\kappa} C_{\kappa}(\lambda F(I + \lambda F)^{-1})/k!] \sum_{d=0}^k \sum_{\delta} a_{\kappa, \tau} C_{\delta}(-\lambda^{-1} \Lambda^{-1})/C_{\delta}(I) \\ \sum_{t=0}^d \sum_{\tau} (-1)^t a_{\delta, \tau} (\frac{1}{2}v_1)_{\tau} C_{\tau}((I + \Omega_1)^{-1} \Omega_1) / [(\frac{1}{2}v_1)_{\tau} C_{\tau}(I)],$$

$$0 < f_1 \leq f_2 \leq \dots \leq f_p < \infty,$$

$$\text{where } \Omega_1 = \Sigma_3^{-\frac{1}{2}} M' \Sigma_1^{-1} M \Sigma_3^{-\frac{1}{2}}.$$

In (9.4) taking $(I + \Omega_1)^{-1} \Omega_1 = p^2$, $\Lambda = I$, $\lambda=1$, $v_3 = v_1+v_2$ and $\tau=\delta=\kappa$, the non-null distributions of Constantine (1963) for III) is obtained.

The non-null distributions of Khatri (1967) for I) and Constantine for II) and III) have been used in the two-roots case to compute exact powers of the criteria 1) to 4) by Pillai and Jayachandran (1967, 1968), of 5) by Sudjana (1973) and of individual ch. roots by Pillai and Dotson (1969) and Pillai and Al-Ani (1970). The expressions in (9.1) have been used by Pillai and Sudjana (1975) to study exact robustness of 1) to 4) against non-normality for I) and against the violation of the assumption of equal covariance matrices

in II). Further, Pillai and Hsu (1975) have used (9.4) to study the exact robustness of 1) to 4) against non-normality for III). Zonal polynomials of degree $k \geq 6$ have been used in the above computations.

In the one-sample case, the joint density of the ch. roots of a Wishart matrix in the non-central case with unknown Σ and Ω has not been worked out. But the density when a) $\Omega = 0$ and b) Σ is known can be obtained as special cases of the following one obtained from the joint density of $B = \text{diag.}$

$(b_1, \dots, b_p) = \lambda F(I + \lambda F)^{-1}$ by putting $G = v_2 B$ and making v_2 tend to infinity:

Theorem 9.3. The joint density of g_1, \dots, g_p as a limiting case from (9.1) is given by (writing v for v_1 and Σ for Λ)

$$(9.5) \quad K_1(p, v) e^{-\text{tr} \Sigma^{-1} \Omega} |\Sigma|^{-\frac{1}{2}v} |G|^{\frac{1}{2}(v-p-1)} e^{-\text{tr} G} \prod_{i>j} (g_i - g_j) \\ \sum_{k=0}^{\infty} \sum_{\kappa} [C_{\kappa}(G)/k!] \sum_{d=0}^k \sum_{\delta} a_{\kappa, \delta} C_{\delta}(-\lambda^{-1} \Sigma^{-1}) L_{\delta}^{\frac{1}{2}(v-p-1)}(G) / (\frac{1}{2}v)_{\delta} C_{\delta}^2(I),$$

$$0 < g_1 \leq \dots \leq g_p < \infty,$$

where $K_1(p, v) = \pi^{\frac{1}{2}p^2} / [\Gamma_p(\frac{1}{2}v) \Gamma_p(\frac{1}{2}p)]$.

Note that $K_1(p, v) = K(p, m)$ expressed in different notations. Formula (9.5) yields the following special cases:

a) $\Omega = 0$. As in (9.1) when $\Omega = 0$ applying (9.3), the form of the density given by James (1960) and Pillai and Al-Ani (Pillai and Al-Ani, 1967, Pillai, Al-Ani and Jouris, 1969) is obtained.

b) Σ known. Again, as in (9.1), putting $\Sigma = I$, $\lambda=1$, using the relation (9.2) and $\tau=\delta=\kappa$, the form given by James (1961, 1964) is deduced.

In the derivation of the densities above for the ch. roots, there are two results which are very important and are given below.

Lemma 9.1. Let $R(p \times p)$ be a positive definite symmetric (pds) matrix and $H(p \times p)$ be an orthogonal matrix with positive first column such that $H'RH = \text{diag}(r_1, \dots, r_p)$, where r_1, \dots, r_p are the ch. roots of R . Under this transformation the volume element dR becomes (James, 1954)

$$(9.6) \quad dR = \prod_{i>j} (r_i - r_j) \prod_{i=1}^p dr_i dH$$

where the measure dH is that derived by the exterior product of differential forms on the orthogonal group. With this measure $\int_{O(p)} dH = 2^p \pi^{\frac{1}{2}p^2} / \Gamma_p(\frac{1}{2}p)$,

where $O(p)$ denotes the group of all orthogonal $(p \times p)$ matrices, (James, 1954).

Lemma 9.2 Let S be a symmetric matrix and R , a pds matrix. Then (James, 1960)

$$(9.7) \int_{O(p)} C_K(H'SH R)(dH) = C_K(S)C_K(R)/C_K(I),$$

where (dH) is the invariant Haar measure on the orthogonal group, normalized so that the measure of the whole group is unity.

In the special case (b) of (9.1), the use of the above two lemmas will give the densities of $F = S_2^{-\frac{1}{2}} S_1 S_2^{-\frac{1}{2}}$ and $B = (S_1 + S_2)^{-\frac{1}{2}} S_1 (S_1 + S_2)^{-\frac{1}{2}}$ as below.

Theorem 9.4 The density of F is given by (James, 1964)

$$(9.8) C_1(p, v_1, v_2) e^{-\text{tr} \Omega} |F|^m |I+F|^{-\frac{1}{2}(v_1+v_2)} {}_1F_1\left(\frac{1}{2}(v_1+v_2); \frac{1}{2}v_1; \Omega F(I+F)^{-1}\right)$$

and the density of B is given by

$$(9.9) C_1(p, v_1, v_2) e^{-\text{tr} \Omega} |B|^m |I-B|^n {}_1F_1\left(\frac{1}{2}(v_1+v_2); \frac{1}{2}v_1; \Omega B\right),$$

where $n = \frac{1}{2}(v_2 - p - 1)$, $C_1(p, v_1, v_2) = \Gamma_p\left[\frac{1}{2}(v_1+v_2)\right] / [\Gamma_p\left(\frac{1}{2}v_1\right) \Gamma_p\left(\frac{1}{2}v_2\right)]$

and hypergeometric function of matrix argument (James, 1964, Constantine, 1963)

$${}_pF_q(c_1, \dots, c_p; d_1, \dots, d_q; A) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(c_1)_{\kappa} \dots (c_p)_{\kappa}}{(d_1)_{\kappa} \dots (d_q)_{\kappa}} \frac{C_{\kappa}(A)}{k!}.$$

The density (9.8) is known as the non-central multivariate F distribution and (9.9) non-central multivariate Beta distribution. Kabe (1963) derived (9.9) when rank of Ω is 2 and Sitgreaves (1952) for the two roots case.

If $v_1 < p \leq v_2$ instead of $p \leq v_1, v_2$ as assumed earlier, the densities in this case can be derived from those above by making the following substitutions:

$$(9.10) (v_1, v_2, p) \rightarrow (p, v_1+v_2-p, v_1).$$

10. SUB-ASYMPTOTIC DISTRIBUTIONS OF THE CH. ROOTS

In recent studies on sub-asymptotic distributions of ch. roots, the approach has generally been to maximize an integral of the form

$$I = \int_{O(p)} {}_pF_q(c_1, \dots, c_p; d_1, \dots, d_q; A H R H') (dH)$$

where $O(p)$ is the group of orthogonal matrices $H(p \times p)$ with respect to which the maximization is carried out, $A = \text{diag}(a_1, \dots, a_p)$, $R = \text{diag}(r_1, \dots, r_p)$, (dH) is the invariant or Haar measure over the group $O(p)$ normalized so that the measure of the whole group is unity, ${}_pF_q$ is a hypergeometric function of matrix argument and $c_1, \dots, c_p, d_1, \dots, d_q$ are functions of df and are positive real numbers. In the one-sample or covariance matrix case, G. A. Anderson (1965)

has shown that the maximum of the integrand ${}_0F_0$ with $p=q=0$, R , the sample ch. root matrix and A^{-1} , the matrix of population ch. roots assumed to be distinct, is attained when $H = \text{diag.}(\pm 1, \dots, \pm 1)$. James (1969) has extended the study to the case of an extreme multiple population root. Chang (1970) and Li and Pillai (Li and Pillai, 1970, Li, Pillai and Chang, 1970) have found the form for H the same as in the one-sample case when maximizing the ${}_1F_0$ integrand in the two-sample (two covariance matrices) problem with $p=1$ and $q=0$. Chattopadhyay and Pillai (1970, Chattopadhyay, Pillai and Li, 1976) have generalized the results of the previous authors not only to ${}_pF_q$ hypergeometric function but also when a_i 's are equal in each of several sets.

The asymptotic expansion for large df is obtained by evaluating the integral for small neighborhoods of the matrices $H = (\pm 1, \dots, \pm 1)$. While the earlier authors considered the asymptotic expansions of a single covariance matrix or two covariance matrices, Chattopadhyay and Pillai (1971b, 1973, Chattopadhyay, Pillai and Li, 1976) have extended the study to MANOVA, ${}_1F_1$, and canonical correlation, ${}_2F_1$, cases and from one extreme multiple root situation to that of one multiple root, extreme or intermediate. In extending the work further to the case of several multiple population roots, the method used by James (1969) was not found to be suitable in view of the fact that the invariance of a function with respect to the choice of a submatrix in the orthogonal matrix used there, does not extend to the simultaneous invariance with respect to choices of several submatrices as is needed to extend the method. Chattopadhyay and Pillai (1971, 1973, Chattopadhyay, Pillai and Li, 1976) have shown that the integrand, if $A = \text{diag.}(a_1, \dots, a_r, a_{r+1}, \dots, a_{r+1}, \dots, a_{r+d}, \dots, a_{r+d}, a_{r+d}, \dots, a_{r+d})$ is optimum if $H = \text{diag.}(\bar{I}_0(r), H_1, \dots, H_d)$, where $\bar{I}_0(r) = \text{diag.}(\pm 1, \dots, \pm 1)$, $H_i(t_i x t_i)$, $i=1, \dots, d$, are orthogonal matrices. It is maximum or minimum depending on the ordering of a_i 's.

The method of expansion is to consider the transformation $H = \exp[S]$ where $S(p \times p)$ is a skew symmetric matrix of the form

$$S = \begin{pmatrix} S^{(0)} \\ S^{(1)} \\ S^{(2)} \\ S^{(d)} \end{pmatrix}, \quad S^{(0)}(r \times p); S^{(i)} = (S_1^{(i)}, S_2^{(i)}, S_3^{(i)}), \quad S_1^{(i)}(t_i x (r+t_1+\dots+t_{i-1})),$$

$$S_2^{(i)}(t_i x t_i) = 0, \quad S_3^{(i)}(t_i x (t_{i+1}+\dots+t_d)), \quad i=1, \dots, d-1;$$

and $S^{(d)} = (S_1^{(d)}, S_2^{(d)}), \quad S_1^{(d)}(t_d x (p-t_d)), \quad S_2^{(d)}(t_d x t_d) = 0.$

With the appropriate ordering of a_i 's, the integrand is maximized when Π is of the above form and for large v_2 , the whole integral is concentrated around its unique maximum. Hence there is no loss of generality in using the above transformation provided the constant factor is adjusted as at least one maximizing set is covered by this substitution. (See Pillai and Chattopadhyay, 1970, 1973). For large v_2 and r_i 's and a_i 's well spaced, most of the integrand will be given by small values of S and terms of the expansion can be computed in view of the transformation $\Pi = \exp[S]$.

Constantine and Muirhead (1976) have very recently studied the asymptotic expansion for the distribution of the ch. roots in the one-sample case for unknown Σ and $\Omega=0$, two-sample F-case for I) and type I case for II) for ch. roots of Ω large. Their method takes the above sub-asymptotic approach as starting point to derive the first term of the expansions but then use a partial differential equation approach (Muirhead 1970a, Constantine and Muirhead, 1972) to successively calculate the other terms up to $O(v^{-3})$, $O(v_1+v_2)^{-3}$ and $O(v_1+v_2)^{-2}$ respectively.

In the one-sample case, Muirhead and Chikuse (1975) have obtained an asymptotic expansion in terms of normal density, of $x_i = (v/2)^{\frac{1}{2}}(t_i a_i - 1)$ $i=1, \dots, p$, using the sub-asymptotic expansion of Anderson (1965), where a_i 's are the ch. roots of Σ^{-1} . The terms up to order v^{-1} have been obtained.

Bingham (1972) has employed a parametrization of the rotation group $O^+(p)$ of $p \times p$ orthogonal matrices with determinant +1 in terms of their skew symmetric parts to derive for $p=3$ an explicit expansion for ${}_0F_0(Z, \Omega)$, a hypergeometric function of two matrix arguments appearing in the distribution of the ch. roots of a $p \times p$ Wishart matrix. On the basis of a numerically derived simplification of the low order terms of this series, an asymptotic expansion for $p=3$ of ${}_0F_0$ in terms of products of ordinary confluent hypergeometric series is conjectured. Limited numerical exploration through terms of order v^{-8} has indicated that the new series is several orders of magnitude more accurate than the series from which it was derived.

11. INDIVIDUAL CH. ROOTS

The study of individual ch. roots in the non-central case has been carried out by several authors. Zonal polynomial series have been useful here. Pillai (1966c, 1970) has obtained the exact cdf of the largest root for II) and III) for the two-roots case and three-roots linear case and computed powers of the test. Pillai and Jayachandran (1967) have obtained an approximation to the largest root in two-, three- and four-roots linear

cases. Again, Pillai and Jayachandran (1968) have obtained the cdf in the two-roots case and powers computed. Pillai and Dotson (1969) have obtained the cdf of the individual ch. roots for II) and III) for two and three-roots cases and Pillai and Al-Ani (1970) for I) and power computed in all cases. In all the above study the zonal polynomials up to sixth degree have been used. Al-Ani (1970) has obtained the distributions of the second largest roots for I) II) and III) in terms of zonal polynomials. Further, Pillai and Al-Ani (1969) have obtained the distributions of the smallest and second smallest roots for I), II) and III) and the one-sample case. In the MANOVA case i.e. hypothesis II), Hayakawa (1967) and independently Khatri and Pillai (1968a) have obtained the density of the largest root in a beta function series form. The density involves two types of coefficients, $g_{\kappa, \tau}^{\delta}$ and $b_{\kappa, \tau}$ seen from the following relations:

$$(11.1) \quad C_{\kappa}(\underline{A})C_{\tau}(\underline{A}) = \sum_{\delta} g_{\kappa, \tau}^{\delta} C_{\delta}(\underline{A}),$$

where $\kappa = (k_1, \dots, k_p)$, $\tau = (t_1, \dots, t_p)$ and $\delta = (d_1, \dots, d_p)$ such that $d = k + \tau$, where κ is a partition of k , τ of t and δ of d . Further, if $M_1 = \text{diag}(M, 1)$, where $M = \text{diag}(m_1, \dots, m_{p-1})$, m_i 's being the ch. roots of M , then $C_{\kappa}(M_1)$ could be written as

$$(11.2) \quad C_{\kappa}(M_1) = \sum_{t=0}^k \sum_{\tau} b_{\kappa, \tau} C_{\tau}(M).$$

Khatri and Pillai (1968a) have tabulated $g_{\kappa, \tau}^{\delta}$ coefficients for $d \leq 7$ and $b_{\kappa, \tau}$ coefficients for $k \leq 6$. Hayakawa (1967) has tabulated $h_{\kappa, \tau}^{\delta}$ coefficients which are $g_{\kappa, \tau}^{\delta}$ times a factor, for $d \leq 4$. Similarly Hayakawa has $a_{\kappa, \tau}$ coefficients corresponding to Khatri and Pillai's $b_{\kappa, \tau}$'s. Further Khatri (1967) has obtained the density of the largest root for I) as a ${}_3F_2$ hypergeometric function.

ai and Sugiyama (1969) have obtained simpler power series expressions for the density and the cdf of the largest root for the MANOVA case than obtained before and also derived those of the largest root for test of I) and III) both in the type I and type II cases. De Waal (1969a) has also considered the distribution of the largest root for III). Further, Troskie and Money (1972) have obtained the distribution of the smallest root for III). Waikar (1973) also obtains the joint density of f_1 and f_p as well as g_1 and g_p . Khatri (1972) has obtained the distribution of the largest as well as the smallest roots for I), II) and III) in exact finite series form involving zonal polynomials when $n - \frac{1}{2}(v_2 - p - 1)$ is a non-negative integer. Based on Khatri's finite series, Venables (1973) has presented an algorithm for the numerical evaluation of the null distribution of b_p . Krishnaiah and Chattopadhyay (1975) have discussed the extensions of the

pfaffian method to the non-central case.

In the one-sample case, Hayakawa (1969) has obtained the density of the largest root of the non-central Wishart matrix with known Σ as a series in Hermite polynomials. For unknown Σ but $\Omega=0$, Khatri (1972) has given the distribution of the largest root as well as the smallest root in finite series of zonal polynomials for non-negative integral values of m .

The non-central distributions of Khatri for I), II) or III) or one-sample case although in finite series form is of limited use since they are generally extremely complex. Hence Pillai and Saweris (1974b) have obtained new forms based on Khatri's results, as finite or infinite series, but useful for further work and generally more rapidly convergent than those obtained earlier by Pillai and Sugiyama (1969). They have also given some approximations to these distributions.

In regard to asymptotic distributions, Hsu (1941a) has obtained for II) the following theorem:

Theorem 11.1. Let $z_i = N^{\frac{1}{2}}(2w_h^2 + 4w_h)^{-\frac{1}{2}}(f_i - w_h)$, $(i=m_{h-1}+1, \dots, m_h; h=1, \dots, t)$, $z_i = Nf_i$,

$(i=r+1, \dots, s)$. Then the limiting distribution of z_1, \dots, z_s as the sample sizes tend to infinity is given by the density (sample ch. roots ordered in descending order of magnitude)

$$f(z_1, \dots, z_{m_1}) \dots f(z_{m_{t-1}+1}, \dots, z_r) f_1(z_{r+1}, \dots, z_s)$$

$$\text{where } f(x_1, \dots, x_u) = 2^{-\frac{1}{2}u} \left(\prod_{i=1}^u \Gamma(\frac{1}{2}) \right)^{-1} \prod_{i < j}^u (x_i - x_j) e^{-\frac{1}{2} \sum_{i=1}^u x_i^2},$$

$$u > x_1 \geq \dots \geq x_u > -\infty,$$

$$f_1(z_{r+1}, \dots, z_s) = 2^{-\frac{1}{2}(p-r)(l-1-r)} \prod_{i=1}^{s-r} \Gamma(\frac{1}{2}q - \frac{1}{2}r - \frac{1}{2}i + \frac{1}{2}) \Gamma(\frac{1}{2}i)^{-1}$$

$$\prod_{i=r+1}^s \prod_{j=i+1}^s (z_i - z_j) \prod_{i=r+1}^s z_i^{\frac{1}{2}(q-s-1)} e^{-\frac{1}{2} \sum_{i=r+1}^s z_i}, \quad u > z_{r+1} \geq \dots \geq z_s > 0,$$

where $w_1 > \dots > w_t > 0$ are the ch. roots of Ω of multiplicity j_1, \dots, j_t respectively and $m_0=0$, $m_1=j_1$, $m_h=j_1 + \dots + j_h$ ($h=1, \dots, t$), $r=m_t$ and $q=\max(p, l-1)$ and N is the total of l sample sizes.

Hsu (1941b) has obtained similar asymptotic distributions for III) namely canonical correlation. Further, Anderson (1948) has adopted Hsu's proofs to obtain the limiting distributions of the ch. roots in the regression problem (linear hypothesis) and also that of the ch. roots of a sample covariance

matrix in the non-central case. In all cases, it is interesting to note that for different h the ch. roots are asymptotically independently distributed.

Anderson has further studied (1951, 1963) in the one-sample case the limiting distribution of the ch. roots and vectors in view of principal component analysis. If the characteristic roots of Σ are different, the deviations of the ch. roots of the sample covariance matrix from the corresponding population quantities are asymptotically independently normally distributed with variance $2\lambda_i^2/v, i=1, \dots, p$. Further, when some of the ch. roots of Σ are equal, the asymptotic distribution of sample ch. roots and vectors also have been studied.

Muirhead (1970b) obtained an asymptotic expansion up to order v^{-2} for the largest root in the one-sample case based on his expansion for ${}_1F_1$ function. But it is of limited interest because it is valid only over the range of values less than the smallest ch. root of Σ . Further, Muirhead and Chiguse (1975) using the system of p.d.e's of Muirhead (1970a) have derived an asymptotic expansion up to order v^{-1} of the distribution of $x_p = (v/2)^{1/2}(\lambda_p - 1)$ and similarly that of x_1 . They have also obtained the marginal density of x_1 from the asymptotic joint distribution discussed in the previous section in terms of normal densities which agrees with the result of Sugiura (1973).

12. HOTELLING'S TRACE

The non-central distribution of Hotelling's trace has been obtained by Constantine (1966) through inverse Laplace transform in the following form:

$$(12.1) \quad C_3(p, v_1, v_2) (U^{(p)})^{\frac{1}{2}pv_1-1} \sum_{k=0}^{\frac{1}{2}pv_1-1} \left[\frac{(-U^{(p)})^k}{(\frac{1}{2}pv_1)_k k!} \right] \sum_{\kappa} \left(\frac{1}{2}(v_1+v_2) \right)_{\kappa} L_{\kappa}^m(\Omega),$$

where $C_3(p, v_1, v_2) = C_2(p, v_1, v_2) \Gamma_p(\frac{1}{2}v_1) / \Gamma(\frac{1}{2}pv_1)$,

$m = \frac{1}{2}(v_1 - p - 1)$, $L_{\kappa}^Y(\Omega)$ is the generalized Laguerre polynomial given in (9.2) and $p \leq v_1, v_2$. The series is convergent for $|U^{(p)}| < 1$. The density of $U^{(p)}$ for $v_1 < p \leq v_2$ can be obtained from (2.1) using (9.10). The central case can be obtained by putting $\Omega=0$ as in special case (a) of (9.1).

Pillai (1973, Pillai and Sudjana, 1974) has used (12.1) to suggest the following form for the density of $U^{(p)}$:

$$(12.2) \quad C_3(p, v_1, v_2) (U^{(p)})^{\frac{1}{2}pv_1-1} / (1+U^{(p)}/p)^{\frac{1}{2}p(v_1+v_2)} \sum_{k=0}^{\infty} \{ (-1)^k E_k (U^{(p)}/p)^k / (1+U^{(p)}/p)^k \},$$

where $E_k = [p^k / (\frac{1}{2}pv_1)_k k!] \sum_{\kappa} \left(\frac{1}{2}(v_1+v_2) \right)_{\kappa} L_{\kappa}^m(\Omega) - \sum_{r=0}^{k-1} \left[\prod_{j=r}^{k-1} \left(\frac{1}{2}(v_1+v_2)p+j \right) / (k-r)! \right] E_r, E_1=0$.

For $p=2$ and $\eta=0$, (12.2) reduces to the exact form given by Hotelling (Constantine, 1966). For higher values of p no verification has been possible.

The exact distribution of $U^{(2)}$ using up to the sixth degree zonal polynomial has been obtained by Pillai and Jayachandran (1967, 1968) and used for computing powers of the appropriate tests. Khatri (1967) has obtained the distribution of Hotelling's trace for I) following Constantine (1966) in series form again convergent for $|U^{(p)}| < 1$. Pillai and Sudjana (1965) have further extended the results of Constantine and Khatri into a single density function starting from (9.1), Pillai and Hsu (1975) for canonical correlation using (9.4) and specialized for the two-roots case following Pillai and Jayachandran, for robustness studies against non-normality for I) and III) and against the violation of the assumption of equality of covariance matrices for II).

In regard to the study of moments for II), Hsu (1940a) has obtained the first two moments of $U^{(p)}$ although the second moment is slightly in error as shown by Khatri and Pillai (1967) who have obtained the first four moments. Earlier, Ghosh (1963) has obtained the first moment of $U^{(p)}$ and the variance for values of $p=3$ and 4. Pillai (1964c, 1966c) and Khatri and Pillai (1965, 1966) have obtained the first four moments of $U^{(p)}$ in the linear case based on independent beta variables which extend to the linear case and an approximation to its distribution suggested based on the first three moments. Further, from the study of the first four moments, Khatri and Pillai (1968b) have given two approximations to the distribution of $U^{(p)}$ in the general non-central case. Constantine (1966) has derived the general moment of $U^{(p)}$ in the form

$$(12.3) \quad E(U^{(p)})^k = (-1)^k \sum_{\kappa} L_{\kappa}^m(-u) / (2(p+1-v_2))^{\nu}, \nu_2 > 2k+p-1.$$

In regard to asymptotic distributions of $U^{(p)}$ considerable work has been carried out by various authors. Asymptotic expansion of the distribution of Hotelling's $T_0^2 = v_2 U^{(p)}$ has been obtained by Ito (1956) in the null case for II) up to order v_2^{-2} . The non-null distribution was given by Siotani (1957) and later by Ito (1960) up to order v_2^{-1} . Further, Siotani (1968, 1971) extended his result up to order v_2^{-2} . The above authors generally used Taylor expansion and perturbation techniques of James (1954) either on the cdf or characteristic functions. Fujikoshi (1970) has given two methods of obtaining the non-null distribution of T_0^2 up to order v_2^{-2} by using hypergeometric function and Laguerre polynomial of matrix argument using characteristic function and Laplace transform.

Both lead to the same result which is the same as that of Siotani (1968). The above expansions all are generally in terms of non-central chi-squared variables. Further, Muirhead (1972b) has also derived an expansion up to order v_2^{-1} which can be readily utilized to give further terms in the expansion and which may be difficult to obtain by the use of earlier methods. Hayakawa (1972) has also given an asymptotic expansion of the distribution of T_0^2 to order v_2^{-2} by obtaining some formulae for generalized Laguerre polynomials of matrix argument and univariate Laguerre polynomials.

For III), Fujikoshi (1970) has obtained an asymptotic expansion for T_0^2 for testing the hypothesis of independence between two sets of variates under sequence of alternatives converging to the null distribution with a rate of convergence $v_2^{-\gamma} (\gamma \geq 0)$. By utilizing a close relationship between II) and III), the asymptotic expansion of the distribution of T_0^2 up to order v_2^{-2} has been studied for $\gamma=1$. Fujikoshi has also computed the powers using the asymptotic distribution for II) for the three-roots case. Further, Lee (1971a) has extended Ito's result for II) to III). The asymptotic expansions in all the work above are generally involving non-central chi-squared terms.

Again, Sugiura and Nngao (1973) have obtained asymptotic formulae for the distribution of T_0^2 for III) up to order $(v_1+v_2)^{-1}$ in terms of normal distribution function and its derivatives. Method is based on differential operators on symmetric matrix by Siotani (1957), Ito (1960) and others.

For I), if \underline{S}_1 and \underline{S}_2 are defined as in (9.1) except that $\Omega=0$, Chattopadhyay and Pillai (1971a) have obtained asymptotic expansions for cdf and percentile of T_0^2 to order v_2^{-1} . The expansions hold when $\underline{\Sigma}_1 \underline{\Sigma}_2^{-1} = \underline{I} + \underline{E}$ and $|ch_i \underline{E}| < 1$, $i=1, \dots, p$, where $ch_i(\underline{A})$ denotes the i^{th} ch. root of \underline{A} . Taylor expansion and perturbation techniques have been employed. Pillai and Saweris (1973) have extended the results of Chattopadhyay and Pillai to order v_2^{-2} and also included terms involving $f_{ij} f_{kl}$ in v_2^{-1} order terms which were neglected by the earlier authors where f_{ij} is the (i,j) th element of $\underline{\Sigma}_1 \underline{\Sigma}_2^{-1}$. Further, Pillai and Saweris (1974a) have extended the work of Chattopadhyay (1972) who derived an asymptotic expansion up to order v_2^{-1} for the cdf and percentile of T_0^2 with \underline{S}_1 and \underline{S}_2 as in (9.1) with $\Omega \neq 0$. The extension was to include $f_{ij} f_{kl}$ terms neglected by Chattopadhyay. Hsu and Pillai (1975a) have extended these results further to include terms up to order v_2^{-2} but neglecting terms of higher powers in some Taylor expansions to keep the number of terms to a manageable level for computation. In all this work perturbation technique is used. The asymptotic expansion obtained for the non-central case has been extended also to

the canonical correlation case by making Ω random. Numerical results of powers have been computed from the expansions by Pillai and Saweris (1974) and Hsu and Pillai (1975) for robustness studies for I) and III) against non-normality and II) against the violation of assumption of equality of covariance matrices.

13. PILLAI'S TRACE

The exact distribution of $V^{(p)}$ is more difficult to obtain in the non-central case than even that of $U^{(p)}$ for various obvious reasons and it has not been derived so far. The exact distribution for the two-roots case has been obtained by Pillai and Jayachandran (1967, 1968) for I) II) and III) using up to sixth degree zonal polynomial. Khatri and Pillai (1968a) have obtained the density of $V^{(p)}$ for II) as zonal polynomial series but is convergent only for $|V^{(p)}| < 1$.

In regard to the moments, Pillai (1964c, 1966c) and Khatri and Pillai (1965, 1968a) have obtained the moments of $V^{(p)}$ in the linear case based on the idea of independent beta variables. Again, Khatri and Pillai (1967) have shown that the moments of $V^{(p)}$ can be obtained from moments of $V^{(r)}$ for II) where $r \leq p$ is the rank of Ω . The first two moments of $V^{(2)}$ have been worked out. Further, Pillai (1968) has obtained the mgf of $V^{(p)}$ for I) II) and III) but the mgf of I) has been obtained by Khatri (1967) with a slight error. The results are in zonal polynomial series form which involve the $a_{k,\tau}$ coefficients in (9.3) which have been tabulated only for $k \leq 8$ (Constantine, 1966, Pillai and Jouris, 1969).

Pillai and Sudjana (1975) have extended the work of Pillai and Jayachandran (1967, 1968) for robustness studies for I) and II) using (9.1) and Pillai and Hsu (1975) for III) using (9.4) as in the case of $U^{(p)}$. They have also extended (Pillai and Sudjana, 1972) the mgf of Pillai (1968) starting from the density (9.1) and Pillai and Hsu (1975) starting from (9.4).

In regard to asymptotic distributions, Fujikoshi (1970) has obtained an asymptotic expansion of chi-squared terms for the distribution of $V^{(p)}$ for II) up to order v_2^{-2} , by using Pillai's mgf (1968) and using certain formulae for weighted sums of zonal polynomials. He has shown the usefulness of the asymptotic expansion by obtaining approximate upper 5 and 1% points and comparing with those of Pillai and Jayachandran (1967) and by computing powers of the $V^{(p)}$ test for three-roots case. Similarly for III), Fujikoshi (1970) has obtained an asymptotic expansion using Pillai's mgf. For $\gamma=1$ (see Section 12), the expansion consists of non-central chi-squared terms and has been derived up to order v_2^{-2} . Further, Lee (1971b) has obtained an asymptotic

expansion for $V^{(p)}$ to order v_2^{-3} , and powers computed and upper percentage points of $V^{(p)}$ also computed by Hill-Davis method (1968). Lee (1971a) also has extended to III) the expansion for II). Again Sugiura and Nagao (1972) have obtained asymptotic formulae for the distribution of $V^{(p)}$ for III) up to order $(v_1+v_2)^{-1}$ in terms of normal distribution function and its derivatives using the method of differential operators on symmetric matrix by Siotani (1957), Ito (1960) and others. Fujikoshi (1972) has further obtained an asymptotic expansion to order v_1^{-2} for II) by employing a method similar to that of Sugiura and Nagao (1971). The asymptotic formulae are given in terms of chi-squared variables.

14. WILKS' CRITERION

The exact non-central distribution of Wilks' criterion has been derived by Pillai, Al-Ani and Jouris (1969) for I), II) and III) using inverse Mellin transform in terms of Meijer's G-functions. Pillai and Nagarsenker (1972) has obtained the distributions of a statistic $\prod_{i=1}^p b_i^a (1-b_i)^b$ of which

Pillai-Al-Ani-Jouris results are special cases. Pillai and Sudjana (1975) have extended the results starting from (9.1) and Pillai and Hsu (1975) using (9.4). Earlier, Pillai and Jayachandran (1967, 1968) have obtained the exact distribution of Wilks' criterion for I), II) and III) in the two-roots case and computed powers in each case. Das Gupta (1972) has shown that the null and the non-null linear distribution of Wilks' criterion have the monotone λ_r property. Das Gupta and Perlman (1973) have shown that the power of $W^{(p)}$ strictly decreases with the dimension and v_1 in the linear case.

Further, in view of independent beta variables approach which extends to the linear case with the non-centrality parameter involved only in the density of t_{11}^2 in (6.2) as can be seen from (9.9) (or (9.8)) by transformation as in Section 6, Gupta (1971) obtained the density of Wilks' criterion for II) in the linear case for number of roots 2 to 5 using the convolution technique (see Section 6). Asoh and Okamoto (1969) have expressed the criterion as a product of conditional betas in the non-null case while Kabe and Gupta (1972) have given an explicit procedure for decomposing the non-central multivariate beta density in terms of densities of independent beta variables extending the work of Pillai (1964c, 1966c) and Khatri and Pillai (1965). Again, Hart and Money (1975) have obtained the distribution of Wilks' criterion in the linear case in a general form and have developed an algorithm for calculating exact powers

and percentiles. Also, tabulations of powers and percentiles have been carried out for selected number of variates and df. Tretter and Walster (1975) also have obtained the distribution in the linear case as mixtures of beta distributions. Nagarsenker (1976) has derived the distribution of I), II) and III) in the general case in simpler computable forms as mixtures of beta distributions as well as mixtures of gamma distributions which are more useful than Pillai-Al-Ani-Jouris forms involving G-functions. The G-functions are not easy to evaluate in all cases. Mathai (1970a) has given series expansions for certain cases of G-functions and shown the use of the series for distributions of Wilks' criterion derived by Pillai-Al-Ani-Jouris. Mathai (1973a) has brought about the points involved in the evaluation of the G-function in terms of computable expressions.

In regard to approximate or asymptotic distributions, Ito(1962) has made numerical power comparisons of test of II) based on T_0^2 and Wilks' criterion in large samples in the linear case. Mikhail (1965) has given such comparison by an approximate method using the lower order moments based on $U^{(2)}$, $V^{(2)}$ and $W^{(2)}$. Further, Posten and Bergmann (1964) have given asymptotic expansions for II) using characteristic functions in the linear case and Roy (1966) has also tabulated the power through an approximate method. Sugiura and Fujikoshi (1969) have derived for II) an asymptotic expansion involving chi-squared terms to order v_2^{-2} for Wilks' criterion using the characteristic function expressed in terms of hypergeometric function of matrix argument. Olkin and Siotani (1964) have shown for III) that the limiting distribution of $(n')^{1/2} W^{(p)}$ is normal. For III), the h^{th} moment of the criterion has been worked out first and then studying the characteristic function Sugiura and Fujikoshi (1969) have also obtained an asymptotic expansion up to order $(v_1+v_2)^{-1}$ in terms of normal distribution function and its derivatives. Fujikoshi (1970) has computed approximate powers based on these expressions in the three-roots case. Lee (1971a) has given an asymptotic expansion for III) using Sugiura-Fujikoshi (1969) expansion for II) as also Sugiura (1969). Power comparisons have been made (Lee 1971b) of $U^{(p)}$, $V^{(p)}$ and $W^{(p)}$ for II) in the three-and four-roots cases. Muirhead (1972a) has obtained an asymptotic expansion for the distribution of λ_{rc} up to order $(v_1+v_2)^{-3}$ for local alternatives in terms of non-central chi-squared distribution and density by obtaining a solution of the system of pde's satisfied by the mgf. This extends the work of Sugiura (1969). Pillai and Nagarsenker (1972) has obtained an asymptotic expansion for I) up to order $(v_1+v_2)^{-1}$ in terms of normal distribution function and its derivatives and Subrahmaniam (1975) has

extended the same to order $(v_1 + v_2)^{-3/2}$ using pde's of Muirhead (1970a) giving power tabulations of the test, the actual F test and $Z^{(p)}$ test.

15. OTHER STATISTICS

As in the null case, Wilks' statistic, $Z^{(p)}$, will be considered first in this section. The approach to the distribution problem is generally the same as that for Wilks' criterion but there has been less interest about this study in view of the inferior nature of the tests regarding power compared to the more outstanding test criteria like $U^{(p)}$, $V^{(p)}$ and $W^{(p)}$. De Waal (1968) has derived the non-central density of $Z^{(p)}$ in terms of an integral expression. Pillai and Nagarsenker (1972) have obtained the distribution of a statistic of the form $\prod_{i=1}^p b_i^a (1-b_i)^b$ of which $Z^{(p)}$ is a special case. The distribution has

been derived in terms of H-functions (Mathai, 1970a) as a result of inverse Mellin transform. The exact distribution has been obtained for I), II) and III) and asymptotic expansions have been obtained to order $(v_1 + v_2)^{-1}$ in terms of normal distribution and derivatives for I) under local alternatives. Subrahmaniam (1975) has extended the results to order $(v_1 + v_2)^{-3/2}$. Sudjana (1973b) has obtained the general form of the distribution of $Z^{(p)}$ starting from (9.1) and (9.4) and obtained the distributions for I), II) and III) as special cases. He has also given power tabulations based on detailed study of the two-roots case and compared the powers with those of $U^{(2)}$, $V^{(2)}$, $W^{(2)}$ and largest root. The distribution in the non-central linear case has been studied by Troskie and Money (1974) following Gupta (1971) obtaining the distribution of $Z^{(p)}$ for II) for $p=2,3,4$ and 5. They have also extended the results to III). Further, Hart and Money (1976) have applied their (1975) algorithm for Wilks' criterion with modification to obtain non-central linear percentiles and powers of $Z^{(p)}$ and carried out numerical power studies whose findings agree with those of the extensive study of Sudjana (1973b). De Waal (1968) has obtained an asymptotic distribution for $Z^{(p)}$ assuming that Ω is a fixed matrix. Fujikoshi (1972) has derived an asymptotic expansion up to order v_1^{-3} involving chi-squared terms by inverting the characteristic function expressed in terms of hypergeometric function with matrix argument.

Regarding the three Harmonic mean criteria, $H_i^{(p)}$, $i=1,2,3$, following Constantine (1966) and Khatri and Pillai (1968a), the densities have been studied by Troskie (1971) which are convergent only in the range $|H_i^{(p)}| < 1$. The densities have been obtained in the context of III). Further, Troskie and Money (1972) have obtained the densities for II) and mgf's for II) and III).

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The exact distribution of Bagai's statistic, $\gamma^{(p)}$, has been derived by Pillai and Nagarsenker (1972) in terms of G- and H-functions using inverse Mellin transform for I), II) and III). Sugiura (1969a) has obtained an asymptotic expansion for I) for large sample sizes and local alternatives.

In regard to the lr test of I), it falls as a special case of the criterion of Pillai and Nagarsenker (1972) given above, namely, $\prod_{i=1}^p b_i^a (1-b_i)^b$

with $a = v_1/2$ and $b = v_2/2$. The exact distribution has been given by the authors in terms of G- and H-functions. Further they have obtained asymptotic expansions for the criterion inverting the characteristic function in two forms, the first as chi-squared series up to order $(v_1+v_2)^{-2}$ and the second as chi-squared series up to order m^{-2} where $m=p(v_1+v_2)$ where p is a correction factor (Anderson, 1958, p.255) but both under local alternatives. Subrahmaniam (1975) have extended these expansions using the pde's of Muirhead (1970a) to order $(v_1+v_2)^{-3}$ and m^{-3} respectively.

In the one-sample case, the exact non-central distribution of the L statistic has not received a great deal of attention. Sugiura (1969b) has obtained asymptotic expansion of the distribution of the criterion $-2\log L$ up to order v^{-1} involving normal distribution function by inversion of the characteristic function.

In regard to the sphericity criterion, Girshick (1941) has obtained the non-null distribution of sphericity criterion in the two-roots case in series form. Pillai and Nagarsenker (1971) have obtained the exact distribution in terms of zonal polynomials and G-function by inverse Mellin transform. Khatri and Srivastava (1971) have also derived the distribution in similar form. Steffens (1974) has obtained power tabulations for the two-roots case obtaining an F series form for the distribution equivalent to that of Girshick (Girshick, 1941, James, 1966) and shows that the power of the sphericity test excels that of a bivariate t test of sphericity suggested in the paper except in the close neighborhood of the hypothesis. Muirhead (1976) has also computed powers of W in the two-roots case based on the non-null distribution which he has obtained as mixtures of F distributions similar to that of Steffens.

As regards asymptotic distributions, Gleser (1966) has given a normal asymptotic form and so has Sugiura (1969b). Sugiura (1969b) has given the asymptotic expansion of $-2 \log W$ to order v^{-1} involving normal distribution function and its derivatives by inverting the characteristic function.

Sugiura (1972) has also obtained the limiting distribution of a locally best invariant test discussed by John (1972) and Sugiura (1972) given by $\text{tr } G^2 / (\text{tr } G)^2$.

As regards generalized variance, Mathai (1970a, 1972a) has obtained the exact non-central distribution of the determinant of the SP matrix as series involving zonal polynomials and G-functions obtained through inverse Mellin transform. Further, the G-functions are evaluated in computable series form.

Fujikoshi (1968) has obtained the limiting distribution when $\Omega = \nu \Theta$ and has derived an asymptotic expansion of the distribution of the sample generalized variance to order $\nu^{-3/2}$ when Ω is constant. Further, Fujikoshi (1970) has extended the study to the case $\Omega = \sqrt{\nu} \Theta$ obtaining the expansion up to order $\nu^{-3/2}$ in terms of normal distribution functions and its derivatives by inverting the characteristic function. Sugiura and Nagao (1971) also have obtained a similar expansion.

In regard to elementary symmetric functions, Khatri and Pillai (1968b) have obtained a recurrence relation to find the moments of the i^{th} esf in p b_j 's as well as p f_j 's in the non-central linear case in terms of those of partitioned submatrices. Pillai and Jouris (1969) have given exact expressions for the first two moments of $V_{i,m,n}^{(p)}$ and first three moments of $U_{i,m,n}^{(p)}$ based on the results of Khatri and Pillai (1968b) and also the third moments of the second esf of a matrix in the non-central means case (James, 1961). The first two moments in question have been obtained earlier by Pillai and Gupta (1968) in terms of generalized Laguerre polynomials. The expected values of the esf's in the p ch. roots of a Wishart matrix in the central case has been obtained by De Waal (1972a) and a conjecture in the non-central case which was proved to be correct by Shah and Khatri (1974). The h^{th} moment of the trace of a non-central Wishart matrix with $\Sigma = \sigma^2 I$ has been studied by Nel (1971) and first six moments explicitly computed for general Σ (1972). De Waal (1973) and De Waal and Nel (1973) have further studied the expected values of the esf's in the Wishart and correlation matrices cases. Further, Fujikoshi (1970) has obtained an asymptotic expansion of the distribution of the trace of a non-central Wishart matrix under the assumptions $\Omega = O(1)$ and $O(\nu)$ to order $\nu^{-3/2}$ in terms of normal distribution function and its derivatives by inverting the characteristic function.

16. A SHORT CRITICAL REVIEW

Over the past fifty years, considerable work has been promoted in multivariate distribution theory and a large number of papers written in the area has been devoted to characteristic root distributions. While the null distribution of the ch. roots was obtained as early as 1939, the non-null ch. root distributions were derived only in the sixties and those too in series forms which are not immensely useful in view of convergence difficulties and the inability to compute the general terms. However, even more unsatisfactory has been the study of the exact distributions of the test criteria which are functions of the ch. roots and which have been established to have various optimum properties (see Section 3). The exact distributions of individual roots have been obtained explicitly for small number of roots but when the number of roots becomes larger, one has to resort to approximations. Such approximations have been suggested only for the largest (smallest) roots (see Pillai, 1954, 1956a, 1965, Section 4). Attempts to obtain approximations for other roots have failed (see Davis, 1972a, Section 4). The study of the exact null distribution of the trace statistics has met with less success. Nanda (1950), Pillai and Jayachandran (1970), Davis (1970b), Krishnaiah and Chang (1972), and others have worked on the exact null distribution of $V^{(p)}$ obtaining limited results for small number of roots and small values of arguments m and n . However, what they have been in full agreement is to recommend the use of Pillai's tables (1960) of approximate percentage points to practitioners. While Pillai has given such tabulations for number of roots from 2 to 8, Mijares (1964) has extended the tables up to number of roots 50. Again, the exact null distribution of Hotelling's trace has been investigated by several authors with somewhat disappointing results. Constantine (1966), Pillai and Young (1971), Pillai and Sudjana (1974), Davis (1968, 1970c) and others have worked on the problem but the general non-null distribution has not been obtained. Here again, the practitioner has to resort to Pillai's tables (1960) giving approximate percentage points for number of roots 2 to 8. However, progress has been made in regard to the exact null distribution of Wilks' criterion in view of inverse Mellin transform approach which was introduced by Nair (1938) along with the differential equation approach and theory of residues, which are some of the important approaches in distribution theory followed by several authors (see Section 6).

Further, approximations to the distribution of Wilks' criterion were available in the literature for a long time (Bartlett, 1938, 1947a, Rao, 1948). In addition, the exact null distribution of $Z^{(p)}$ follows that of $W^{(p)}$, and those of other λ_r test criteria, for example, L , sphericity and others also have been obtained through inverse Mellin transforms. The study of the distributions of the determinants, for example, Bagai's statistic, generalized variance, has been facilitated by the use of G-functions (see Section 7).

In regard to the non-null distributions, the exact distribution problem has generally met with disappointing results. The exact distributions of the largest (smallest) or other roots have been considered by many authors (Pillai and Sugiyama, 1969, Al-Ani, 1970, Khatri, 1972, Pillai and Saweris, 1974b and others) in finite or infinite series forms involving zonal polynomials whose general terms cannot be explicitly computed and with slow or questionable convergence in the infinite case. The exact non-null distributions of $V^{(p)}$ or $U^{(p)}$ have not been obtained in the general case. The distributional forms involving zonal polynomial series derived in this connection for $V^{(p)}$ or $U^{(p)}$ are convergent only for $|V^{(p)}|$ or $|U^{(p)}| < 1$ (Constantine, 1966, Khatri and Pillai, 1968a, Khatri, 1967 and others). However, the general moments of $U^{(p)}$ has been obtained (Constantine, 1966, Khatri and Pillai, 1967) and the mgf's of $V^{(p)}$ (Pillai, 1968) all involving zonal polynomials and some constants whose general expressions are unavailable. The exact non-null distributions of Wilks' criterion have been derived in terms of G-functions (Pillai-Al-Ani and Jouris, 1969) using inverse Mellin transform. The G-functions cannot be easily evaluated in all cases. But computable series form expressions for special cases of G-functions involved in the distribution of $W^{(p)}$ and others have been obtained (Mathai, 1970a). Attempts have also been made to give the distribution as mixtures of beta or gamma distributions (Tretter and Walster, 1975, Nagarsenker, 1976). As in the null case, the non-null distribution of $Z^{(p)}$ is similar to that of $W^{(p)}$ and those of Bagai's statistic, sphericity and generalized variance involve G-functions and zonal polynomials.

While a great deal remains yet to be accomplished on the exact null and non-null distributions, extensive work has been carried out on asymptotic distributions. The sub-asymptotic expansions treated in Section 10 are interesting, however the distributional forms obtained are not of readily usable form. Starting with the sub-asymptotic expansions, some asymptotic expansions have been obtained in usable form (Muirhead and Chiguse, 1975, Constantine and Muirhead, 1976, Section 10) but the sub-asymptotic nature has disappeared.

Asymptotic expansions have been obtained in a large number of cases by inverting asymptotic formulae of the characteristic functions or mgf (Ito, 1960, Siotani, 1971, Sugiura and Fujikoshi, 1969, Fujikoshi, 1970, Muirhead, 1970b and others). In one set of expansions, use is made of some formulae for weighted sums of zonal polynomials and generalized Laguerre polynomials (Fujikoshi, 1970, Sugiura, 1971) in another set, Taylor expansions and perturbation techniques are used (Ito, 1956, Siotani, 1971, Chattopadhyay and Pillai, 1971, Pillai and Saweris, 1973, Hsu and Pillai, 1975 and others) and in a third set de's are employed (Davis, 1968, 1970a, 1970b, 1972a, Muirhead, 1970b and others) to obtain solutions for mgf's or density functions. The results have been obtained generally for large sample sizes in terms of chi-squared distribution, central or non-central, or normal distribution. Some expansions in the non-central case are for local alternatives.

The usefulness of these asymptotic results have been very little so far for the practitioner. Some results, like the asymptotic independence and normality of the ch. roots in the one-sample case for principal component analysis, have been very helpful. But then the unanswered question is: How large should the sample size be to use these results?

In conclusion, it has to be recognized that quite a large number of papers have been written in the field of ch. roots distributions employing various methods of approach. Merits and demerits of some of these methods have been discussed by Mathai (1973b). Although considerable efforts have been made by various authors on exact distribution theory especially of different test criteria, the results have not been encouraging. Again, even though the exact non-null distributions of the ch. roots have been available for sometime, it seems desirable to have these obtained in some other more useful forms than the existing ones i.e., series involving zonal polynomials whose convergence is slow or questionable and general terms are not computable. The recent progress made in asymptotic distribution theory remains yet to be appreciated and assimilated by practitioners.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) A review of the work on distributions of characteristic (ch.) roots in real Gaussian multivariate analysis has been attempted surveying the develop- ments in the field from the start covering about fifty years. The exact null and non-null distributions of the ch. roots have been reviewed and sub-asym- ptotic and asymptotic expansions of the distributions mostly for large sample sizes studied by various authors, have been briefly discussed. Such distribu- tional studies of four test criteria and a few less important ones which are functions of ch. roots have further been discussed in view of the power		

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comparisons made in connection with tests of three multivariate hypotheses. In addition, one-sample case has also been considered in terms of distributional aspects of the ch. roots and criteria for tests of two hypotheses on the covariance matrix. A brief critical review has also been attempted. For convenience in organization, the review has been given in two parts: Part I. Null distributions and Part II. Non-null distributions.